A

COLLECTION OF PROBLEMS

IN

ILLUSTRATION OF THE PRINCIPLES

OF

THEORETICAL MECHANICS

BY WILLIAM WALTON, M.A.

TRINITY COLLEGE, CAMBRIDGE,

MATHEMATICAL LECTURER AT MAGDALENE COLLEGE.

"Examples give a quicker impression than arguments."—BACON.

SECOND EDITION.

CAMBRIDGE:

DEIGHTON, BELL AND CO.

LONDON: BELL & DALDY.

1855.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS.

PREFACE TO THE FIRST EDITION.

THE design of this Work is to facilitate the study of Theoretical Mechanics, by presenting to the student a systematic collection of Problems in illustration of the more important principles of the science. The want of any such treatise, it is believed, has been felt by many as a serious impediment to the acquisition of adequate ideas in this branch of mathematical philosophy. Much importance, it may be observed, was attached by the great discoverers of the mechanical theories to the full discussion of numerous problems, as will be evident from a reference to the works of the three Bernoullis, of Leibnitz, and of D'Alembert, and to the beautiful investigations scattered throughout so long a series of volumes of the *St. Petersburgh Transactions*, by the liberal hand of Euler.

The author of this volume has endeavoured, as much as possible, to direct the attention of the student to the original memoirs of which he has so largely availed himself. This he has done, partly, to enable the beginner to obtain more detailed information than is compatible with the nature of this work, on particular questions which may excite an interest in his mind: his chief object, however, has been, to offer every facility to those who have already overcome at least the elementary difficulties of the subject, for acquiring a practical familiarity with the historical development of the science. Although it be admitted that useful and exact knowledge may be obtained from even an exclusive perusal of the concise and methodical treatises which are generally adopted for the purpose of academic instruction; yet it may be asserted with confidence, that an excessive adherence to such a system of study, must deprive the student of much delightful and most valuable information.

In regard to the mode in which the author of this treatise has completed the task which he has proposed to himself, he feels every degree of diffidence, and would willingly that it had been undertaken by an abler hand. In apology for the imperfections, of which either he is himself aware or which may have eluded his observation, he can plead only the fact of engrossing occupations, or of perhaps insufficient preparation for a work requiring greater research than was originally contemplated.

Many of the problems in this volume have been extracted, with appropriate modifications, from the Ancient Transactions of the various Academies and learned Societies of Europe; many have been selected from the Cambridge Senate-House Papers; and for not a few the author is under obligation to the contributions of his friends. In arriving at original sources of information, it is scarcely necessary to state that great assistance has been obtained from the historical matter of *Lagrange's Mécanique Analytique*, and from *Montucla's Histoire des Mathématiques*.

CAMBRIDGE, *October*, 1842.

PREFACE TO THE SECOND EDITION.

IN preparing for the press a Second Edition of this Treatise, the author has adhered to the general design of the First Edition; he has, however, effected numerous alterations and corrections, many of which are due to the kind suggestions of readers of the work; he has also considerably augmented the matter of those chapters which in the former edition appeared to be inadequately supplied with problems. Certain entirely new chapters have also been written; one on the Attractions of Solid Bodies, two on Miscellaneous Problems, and one on Live Things.

CAMBRIDGE, *8th September*, 1855.

CONTENTS.

STATICS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

DYNAMICS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

CHAPTER XIV.

STATICS.

CHAPTER I.

CENTRE OF GRAVITY.

LET dm represent an element of the mass of a body at any point x, y, z, referred to any three co-ordinate axes, rectangular or oblique, and let $\bar{x}$, $\bar{y}$, $\bar{z}$, denote the co-ordinates of the centre of gravity of the body; then the formulæ for finding the values of $\bar{x}$, $\bar{y}$, $\bar{z}$, are

$$\bar{x} = \frac{\int x dm}{\int dm}, \quad \bar{y} = \frac{\int y dm}{\int dm}, \quad \bar{z} = \frac{\int z dm}{\int dm},$$

the limits of the integrations being determined by the form of the body.

If the body be bounded by a surface expressible by a single algebraical equation in x, y, z, the evaluation of each of the expressions $\int x dm$, $\int y dm$, $\int z dm$, $\int dm$, will require the performance of the operation of integration on a single function of x, y, z, between appropriate limits; if, however, the body be bounded by discontinuous surfaces, the evaluation of each of these expressions will require the integration between proper limits of several functions of x, y, z, corresponding to the several discontinuous surfaces; the sum of the definite integrals of these functions being the required value of the expression.

The idea of the centre of gravity of material bodies is due to Archimedes, by whom the centres of gravity of various areas

were investigated in his treatise, entitled 'Επιπέδων ἰσορροπικῶν ἢ κέντρα βαρῶν ἐπιπέδων. He likewise determined the centre of gravity of the parabolic conoid. Among the mathematical successors of Archimedes who have cultivated the science of the centre of gravity, may be mentioned Pappus[1], Guido Ubaldi[2], Lucas Valerius[3], La-Faille[4], Guldin[5], Wallis[6], Carré[7], Varignon[8], Clairaut[9].

SECT. 1. *Symmetrical Area.*

Let x be the abscissa and y the ordinate of any point in the circumference of a plane area, symmetrical with respect to the axis of x; the axes of co-ordinates being either rectangular or oblique. Then the centre of gravity of any portion of this area intercepted between any assigned pair of double ordinates will lie in the axis of x, and its distance $\bar{x}$ from the origin will be given by the formula

$$\bar{x} = \frac{\int xy\, dx}{\int y\, dx},$$

where the integrations are to be performed between limits depending upon the positions of the intercepting ordinates.

The value of $\bar{x}$ is sometimes more readily obtained by polar co-ordinates, when the formula will be

$$\bar{x} = \frac{\iint r^2 \cos\theta\, d\theta\, dr}{\iint r\, d\theta\, dr},$$

where r denotes the distance of any point within the area from

[1] *Mathemat. Collect.*, lib. 8, *published for the first time in* 1588.
[2] *In duos Archimedis Æquiponderantium libros Paraphrasis*, 1588.
[3] *De Centro Gravitatis Solidorum*, 1604.
[4] *De Centro Gravitatis partium Circuli et Ellipsis Theoremata*, 1632.
[5] *Centrobaryca*, 1635.
[6] *Opera*, tom. I. cap. 4 et 5, 1670.
[7] *Mesure des Surfaces*, 1700.
[8] *Mém. de l Acad. des Sciences de Paris*, 1714.
[9] *Mém. de l'Acad. des Sciences de Paris*, 1731, p. 159.

the origin, and θ the inclination of r to the axis of x. The nature of the limits in the double integrations will depend upon the form of the area in each particular case.

Supposing the area to consist of several portions, the boundaries of which are defined by distinct equations, the above formulæ must be replaced by

$$\bar{x} = \frac{\Sigma \int xy\,dx}{\Sigma \int y\,dx},$$

$$\bar{x} = \frac{\Sigma \iint r^2 \cos\theta\,d\theta\,dr}{\Sigma \iint r\,d\theta\,dr},$$

where Σ represents the summation of the integrations performed in regard to the several portions of the area.

(1) To find the centre of gravity of the area of any portion BAC (fig. 1) of a parabola cut off by any chord BC.

Let Py be a tangent to the parabola at a point P parallel to the chord CB; from P draw Px parallel to the axis of the parabola. Then, Px and Py being taken as the axes of x and y, the equation to the curve will be

$$y^2 = 4mx,$$

m being the distance of the point P from the focus.

Hence, if $PE = a$,

$$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx} = \frac{\int_0^a x^{\frac{3}{2}}\,dx}{\int_0^a x^{\frac{1}{2}}\,dx} = \frac{\frac{2}{5}a^{\frac{5}{2}}}{\frac{2}{3}a^{\frac{3}{2}}} = \frac{3}{5}a.$$

Archimedes, Ἐπιπέδων ἰσοῤῥοπικῶν, Lib. II. Prop. 8; Guldin, *Centrobaryca*, Lib. I. cap. 9, p. 121.

(2) To find the centre of gravity of the area of the Cissoid of Diocles, EAE', (fig. 2).

The equation to the curve is

$$y^2 = \frac{x^3}{a-x};$$

hence
$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} = \frac{\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx}{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx} \;\ldots\ldots\ldots\ldots\ldots\; (1);$$

but
$$\int \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = -\,2x^{\frac{5}{2}}\,(a-x)^{\frac{1}{2}} + 5\int x^{\frac{3}{2}}\,(a-x)^{\frac{1}{2}}\,dx,$$

and therefore
$$\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = 5\int_0^a x^{\frac{3}{2}}\,(a-x)^{\frac{1}{2}}\,dx$$

$$= 5a\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx - 5\int_0^a \frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}\,dx = \tfrac{5}{6}a\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx;$$

hence from (1) we have

$$\bar{x} = \tfrac{5}{6}a\,\frac{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx}{\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx} = \tfrac{5}{6}a.$$

(3) To find the centre of gravity of the sector ABC (fig. 3) of a circle, of which C is the centre.

From C draw the straight line CEx bisecting the sectorial area; and draw Cy at right angles to Cx. Let $CE = a$, and $\angle\, ACx = \alpha$; then, Cx, Cy, being the axes of x and y,

$$\bar{x} = \frac{\Sigma\int xy\,dx}{\Sigma\int y\,dx} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

Now the equations to the straight line CA, and to the circle of which AEB is an arc, are respectively

$$y = x\tan\alpha, \quad y^2 = a^2 - x^2;$$

also CF is equal to $a \cos \alpha$; hence

$$\Sigma \int xy\,dx = \int_0^{a\cos\alpha} \tan\alpha\, x^2 dx + \int_{a\cos\alpha}^{a} x\,(a^2 - x^2)^{\frac{1}{2}}\,dx \ldots\ldots. (2),$$

and $$\Sigma \int y\,dx = \int_0^{a\cos\alpha} \tan\alpha\, x\,dx + \int_{a\cos\alpha}^{a} (a^2 - x^2)^{\frac{1}{2}}\,dx \ldots\ldots.. (3).$$

Now, by the ordinary processes of the Integral Calculus,

$$\int_0^{a\cos\alpha} \tan\alpha\, x^2 dx = \tfrac{1}{3} a^3 \sin\alpha \cos^2\alpha,$$

and $$\int_{a\cos\alpha}^{a} x\,(a^2 - x^2)^{\frac{1}{2}}\,dx = \tfrac{1}{3} a^3 \sin^3\alpha\,;$$

hence from (2) we have

$$\Sigma \int xy\,dx = \tfrac{1}{3} a^3 \sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, $$\int_0^{a\cos\alpha} \tan\alpha\, x\,dx = \tfrac{1}{2} a^2 \sin\alpha \cos\alpha,$$

and $$\int_{a\cos\alpha}^{a} (a^2 - x^2)^{\frac{1}{2}}\,dx = \tfrac{1}{2}(a^2\alpha - a^2 \sin\alpha\cos\alpha)\,;$$

hence from (3) we have

$$\Sigma \int y\,dx = \tfrac{1}{2} a^2 \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From the relations (1), (4), (5),

$$\bar{x} = \frac{\tfrac{1}{3} a^3 \sin\alpha}{\tfrac{1}{2} a^2 \alpha} = \tfrac{2}{3} a \frac{\sin\alpha}{\alpha}.$$

This result however may be obtained more readily by polar co-ordinates: let P be any point in the area of the sector; $CP = r$, $\angle PCx = \theta$; then

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha}\int_0^{a} r^2 \cos\theta\, d\theta\, dr}{\int_{-\alpha}^{+\alpha}\int_0^{a} r\, d\theta\, dr} = \frac{\tfrac{1}{3} a^3 \int_{-\alpha}^{+\alpha} \cos\theta\, d\theta}{\tfrac{1}{2} a^2 \int_{-\alpha}^{+\alpha} d\theta} = \tfrac{2}{3} a \frac{\sin\alpha}{\alpha}.$$

We might have effected the double integration in a different order; thus

$$\bar{x} = \frac{\int_0^a \int_{-\alpha}^{+\alpha} r^2 \cos\theta \, dr \, d\theta}{\int_0^a \int_{-\alpha}^{+\alpha} r \, dr \, d\theta} = \frac{2 \sin\alpha \int_0^a r^2 \, dr}{2\alpha \int_0^a r \, dr}$$

$$= \frac{2 \sin\alpha . \frac{1}{3} a^3}{2\alpha . \frac{1}{2} a^2} = \frac{2}{3} a \frac{\sin\alpha}{\alpha} = \frac{2}{3} \frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

According to the former order of integration, the sector ACB is conceived to be subdivided into an infinite series of infinitesimal triangles having a common vertex C, their bases being elements of the arc AEB; according to the latter order, we conceive the sector to be made up of a series of circular rings of indefinitely small breadth, having a common centre C.

Carré; *Mesure des Surfaces*, &c. p. 76.

(4) To find the centre of gravity of the segment $AEBF$ (fig. 3) of a circle.

The construction and notation remaining the same as in the preceding example, produce CP to cut the chord AB in Q and the arc AEB in R.

Then, if $CQ = r'$,

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr}{\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r \, d\theta \, dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, since $r' = a \dfrac{\cos\alpha}{\cos\theta}$, we have

$$\int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr = \tfrac{1}{3}(a^3 - r'^3) \cos\theta \, d\theta = \tfrac{1}{3} a^3 \left(1 - \frac{\cos^3\alpha}{\cos^3\theta}\right) \cos\theta \, d\theta;$$

hence $$\int_{-\alpha}^{+\alpha} \int_{r'}^{a} r^2 \cos\theta \, d\theta \, dr = \tfrac{1}{3} a^3 \int_{-\alpha}^{+\alpha} \left(\cos\theta - \frac{\cos^3\alpha}{\cos^2\theta}\right) d\theta$$

$$= \tfrac{1}{3} a^3 (\sin\theta - \cos^3\alpha \tan\theta), \quad \text{from } \theta = -\alpha \text{ to } \theta = +\alpha,$$

$$= \tfrac{2}{3} a^3 (\sin\alpha - \cos^2\alpha \sin\alpha) = \tfrac{2}{3} a^3 \sin^3\alpha \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, $\displaystyle\int_{r'}^{a} r\,d\theta\,dr = \tfrac{1}{2}(a^2 - r'^2)\,d\theta = \tfrac{1}{2}a^2\left(1 - \frac{\cos^2\alpha}{\cos^2\theta}\right)d\theta,$

and therefore

$$\int_{-\alpha}^{+\alpha}\int_{r'}^{a} r\,d\theta\,dr = \tfrac{1}{2}a^2(\theta - \cos^2\alpha\tan\theta),\quad \text{from } \theta = -\alpha \text{ to } \theta = +\alpha,$$
$$= a^2(\alpha - \sin\alpha\cos\alpha)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

Hence, from (1), (2), (3), we get

$$\bar{x} = \tfrac{2}{3}a\,\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}.$$

This result may be obtained as easily by rectangular co-ordinates; thus, putting $a\cos\alpha = a'$,

$$\bar{x} = \frac{\displaystyle\int_{a'}^{a} xy\,dx}{\displaystyle\int_{a'}^{a} y\,dx}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4);$$

but $\qquad y^2 = a^2 - x^2;$

hence $\qquad \displaystyle\int_{a'}^{a} xy\,dx = \int_{a'}^{a} x(a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}},\ \text{between limits},\ = \tfrac{1}{3}a^3\sin^3\alpha\ \ldots\ldots\ldots\ (5).$$

Again, $\qquad \displaystyle\int_{a'}^{a} y\,dx = \int_{a'}^{a}(a^2 - x^2)^{\frac{1}{2}}\,dx$

$$= (a^2 - x^2)^{\frac{1}{2}}x + \int x^2(a^2 - x^2)^{-\frac{1}{2}}\,dx,\ \text{between limits},$$

$$= (a^2 - x^2)^{\frac{1}{2}}x + a^2\int_{a'}^{a}\frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} - \int_{a'}^{a}(a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}(a^2 - x^2)^{\frac{1}{2}}x + \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a},\ \text{from } x = a' \text{ to } x = a,$$

$$= -\tfrac{1}{2}a^2\sin\alpha\cos\alpha + \tfrac{1}{2}a^2\alpha = \tfrac{1}{2}a^2(\alpha - \sin\alpha\cos\alpha)\ \ldots\ldots(6).$$

Hence from (4), (5), (6), there results

$$\bar{x} = \tfrac{2}{3}a\,\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}.$$

In the integrations by polar co-ordinates the segmental area is conceived to be made up of frustums of an infinite number of infinitesimal triangles intercepted by the chord AB, C being the common vertex of the triangles, and a series of elements of the arc AEB being their bases; on the other hand, when rectangular co-ordinates are made use of, the segment is conceived to be made up of an infinite number of indefinitely thin parallelograms parallel to the chord AB.

Guldin; *Centrobaryca*, Lib. I. cap. 9, p. 107.

(5) To find the centre of gravity of any portion of a semicubical parabola comprised between the curve and a double ordinate.

The equation to the curve being $ay^2 = x^3$, we shall have

$$\bar{x} = \tfrac{5}{7}x.$$

(6) To find the centre of gravity of the whole area of the curve of which the equation is

$$y^2 = b^2 \frac{a-x}{x}.$$

$$\bar{x} = \tfrac{1}{4}a.$$

(7) To find the centre of gravity of a semi-ellipse, the bisecting line being any diameter.

If the bisecting diameter be taken as the axis of y, and the conjugate diameter as the axis of x, the equation to the ellipse will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

and we shall have $$\bar{x} = \frac{4a}{3\pi}.$$

Guldin; *Centrobaryca*, Lib. I. cap. 9, p. 115.

(8) To find the centre of gravity of a loop of the Lemniscata of James Bernoulli.

The equation to the curve being $r^2 = a^2 \cos 2\theta$, we shall have

$$\bar{x} = \frac{2^{\frac{1}{2}}\pi}{8}a.$$

(9) To find the centre of gravity of the whole area of a cycloid.

The equations to the cycloid being

$$x = a(1 - \cos\theta), \qquad y = a(\theta + \sin\theta),$$

we shall have $\qquad \bar{x} = \frac{7}{6}a.$

Sect. 2. *Area not Symmetrical.*

The formulæ for the determination of the co-ordinates of the centre of gravity of an area not symmetrical with respect to either of the axes, are

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}, \qquad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy};$$

x and y in these expressions are the co-ordinates of any point whatever within the area, and the limits of the double integration will depend upon the form of the bounding curve.

It frequently happens that the method of polar co-ordinates is more convenient for the determination of $\bar{x}$ and $\bar{y}$ than that by rectangular co-ordinates: the formulæ will be

$$\bar{x} = \frac{\iint r^2\cos\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}, \qquad \bar{y} = \frac{\iint r^2\sin\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}.$$

(1) To find the centre of gravity of the area CPD (fig. 4) of an ellipse, where CP, CD, are two conjugate semi-diameters.

If $CP = a$, $CD = b$, and CP, CD, produced indefinitely, be taken as the axes of x, y, the equation to the ellipse will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and for the position of the centre of gravity we have, indicating the limits of integration,

$$\bar{x} = \frac{\int_0^a\int_0^y x\,dx\,dy}{\int_0^a\int_0^y dx\,dy}, \quad \bar{y} = \frac{\int_0^a\int_0^y y\,dx\,dy}{\int_0^a\int_0^y dx\,dy},$$

the value of y in the limit being pm in the figure; in the integration indicated with respect to y, the figure $pqnm$ is considered as being made up of an infinite number of indefinitely small parallelograms $p'q'$; and in the integration indicated with respect to x, the whole figure CPD is conceived to be composed of an infinite number of indefinitely thin figures such as $pqnm$.

$$\int_0^a\int_0^y x\,dx\,dy = \int_0^a xy\,dx = \frac{b}{a}\int_0^a x(a^2 - x^2)^{\frac{1}{2}}\,dx,$$

since the value of y in the limit coincides with the ordinate in the equation (1); hence

$$\int_0^a\int_0^y x\,dx\,dy = -\tfrac{1}{3}\frac{b}{a}(a^2 - x^2)^{\frac{3}{2}}, \text{ from } x = 0 \text{ to } x = a,$$

$$= \tfrac{1}{3}a^2b.$$

Again, $$\int_0^a\int_0^y dx\,dy = \int_0^a y\,dx$$

$$= \frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx = \tfrac{1}{4}\frac{b}{a}\pi a^2 = \tfrac{1}{4}\pi ab.$$

Hence by the general formula for $\bar{x}$ we have

$$\bar{x} = \frac{\frac{1}{3}a^2b}{\frac{1}{4}\pi ab} = \frac{4a}{3\pi}.$$

Again, $$\int_0^a\int_0^y y\,dx\,dy = \tfrac{1}{2}\int_0^a y^2\,dx$$

$$= \tfrac{1}{2}\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \tfrac{1}{2}\frac{b^2}{a^2}(a^3 - \tfrac{1}{3}a^3) = \tfrac{1}{3}ab^2,$$

and therefore $$\bar{y} = \frac{\frac{1}{3}ab^2}{\frac{1}{4}\pi ab} = \frac{4b}{3\pi},$$

a result which might have been foreseen from the value of $\bar{x}$.

Instead of the order of the limits which we have chosen, we might equally well have integrated, first with respect to x and

then with respect to y, when the formulæ for $\bar{x}$ and $\bar{y}$ would have been

$$\bar{x} = \frac{\int_0^b \int_0^x x\,dy\,dx}{\int_0^b \int_0^x dy\,dx}, \quad \bar{y} = \frac{\int_0^b \int_0^x y\,dy\,dx}{\int_0^b \int_0^x dy\,dx}.$$

(2) To find the centre of gravity of the segment $APBp$ (fig. 5) of an ellipse cut off by a quadrantal chord ApB.

Let $CA = a$, $CB = b$, $CM = x$, $PM = y$, $pM = y'$; then the equations to the ellipse and to the chord will be

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2), \quad y' = \frac{b}{a}(a - x) \dots\dots\dots\dots (1).$$

The formula for $\bar{x}$ will be, indicating the limits,

$$\bar{x} = \frac{\int_0^a \int_{y'}^y x\,dx\,dy}{\int_0^a \int_{y'}^y dx\,dy} \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

Now $$\int_0^a \int_{y'}^y x\,dx\,dy = \int_0^a (y - y')\,x\,dx$$

$$= \frac{b}{a}\int_0^a \{(a^2 - x^2)^{\frac{1}{2}} - (a - x)\}\,x\,dx, \text{ by the equations (1),}$$

$$= \frac{b}{a}\{-\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}} - \tfrac{1}{2}ax^2 + \tfrac{1}{3}x^3\}, \quad \text{from } x = 0 \text{ to } x = a,$$

$$= \tfrac{1}{6}a^2 b.$$

Also $$\int_0^a \int_{y'}^y dx\,dy = \int_0^a (y - y')\,dx = \frac{b}{a}\int_0^a \{(a^2 - x^2)^{\frac{1}{2}} - (a - x)\}\,dx$$

$$= \frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx - \frac{b}{a}(a^2 - \tfrac{1}{2}a^2)$$

$$= \frac{b}{a}(\tfrac{1}{4}\pi a^2 - \tfrac{1}{2}a^2) = \tfrac{1}{4}(\pi - 2)\,ab.$$

Hence from (2) $$\bar{x} = \tfrac{2}{3}\frac{a}{\pi - 2}.$$

Similarly we should evidently get

$$\bar{y} = \tfrac{2}{3}\frac{b}{\pi - 2}.$$

(3) To find the centre of gravity of the area KSL (fig. 6) of a parabola, of which S is the focus, and SK, SL, any two radii.

Take S as the origin of co-ordinates; also, A being the vertex of the parabola, let ASx be the axis of x, and Sy at right angles to Sx the axis of y. Let $SP=r$, $\angle ASP=\theta$, $AS=m$. Then for the position of the centre of gravity, if $\angle ASK=\alpha$, $\angle ASL=\beta$,

$$\bar{x}=\frac{\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr}{\int_\alpha^\beta\int_0^r r\,d\theta\,dr},\quad \bar{y}=\frac{\int_\alpha^\beta\int_0^r r^2\sin(\pi-\theta)\,d\theta\,dr}{\int_\alpha^\beta\int_0^r r\,d\theta\,dr}.$$

Now $\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=\frac{1}{3}r^3\cos(\pi-\theta)\,d\theta=-\frac{1}{3}r^3\cos\theta\,d\theta$;

but, by the nature of the parabola,

$$r=\frac{m}{\cos^2\frac{1}{2}\theta};$$

hence $$\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}\,d\theta,$$

and therefore $$\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\int_\alpha^\beta\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}\,d\theta;$$

but $$\frac{\cos\theta}{\cos^6\frac{1}{2}\theta}=\frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}\,\frac{1}{\cos^6\frac{1}{2}\theta}=\frac{1-\tan^2\frac{1}{2}\theta}{\cos^4\frac{1}{2}\theta}=(1-\tan^2\tfrac{1}{2}\theta)\sec^4\tfrac{1}{2}\theta;$$

hence

$$\int_\alpha^\beta\int_0^r r^2\cos(\pi-\theta)\,d\theta\,dr=-\frac{1}{3}m^3\int_\alpha^\beta\sec^2\tfrac{1}{2}\theta\,(1-\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\,d\theta$$

$$=-\frac{1}{3}m^3\int_{\tan\frac{1}{2}\alpha}^{\tan\frac{1}{2}\beta}(1-\tan^4\tfrac{1}{2}\theta)\,2d\tan\tfrac{1}{2}\theta$$

$$=-\frac{2}{3}m^3\{\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha-\tfrac{1}{5}(\tan^5\tfrac{1}{2}\beta-\tan^5\tfrac{1}{2}\alpha)\}.$$

Also $$\int_\alpha^\beta\int_0^r r\,d\theta\,dr=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta=\frac{1}{2}m^2\int_\alpha^\beta\frac{d\theta}{\cos^4\frac{1}{2}\theta}$$

$$=\frac{1}{2}m^2\int_{\tan\frac{1}{2}\alpha}^{\tan\frac{1}{2}\beta}(1+\tan^2\tfrac{1}{2}\theta)\,2d\tan\tfrac{1}{2}\theta$$

$$=m^2\{\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha+\tfrac{1}{3}(\tan^3\tfrac{1}{2}\beta-\tan^3\tfrac{1}{2}\alpha)\}.$$

Hence $\bar{x} = \dfrac{2m}{3} \dfrac{\frac{1}{5}(\tan^5 \frac{1}{2}\beta - \tan^5 \frac{1}{2}\alpha) - (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}$.

Again, $$\int_\alpha^\beta \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr = \tfrac{1}{3}\int_\alpha^\beta r^3 \sin\theta\, d\theta$$

$$= \tfrac{1}{3} m^3 \int_\alpha^\beta \frac{\sin\theta}{\cos^6 \frac{1}{2}\theta}\, d\theta = \tfrac{2}{3} m^3 \int_\alpha^\beta \frac{\sin \frac{1}{2}\theta}{\cos^5 \frac{1}{2}\theta}\, d\theta$$

$$= -\tfrac{4}{3} m^3 \int_{\cos\frac{1}{2}\alpha}^{\cos\frac{1}{2}\beta} \frac{d\cos\frac{1}{2}\theta}{\cos^5 \frac{1}{2}\theta} = \tfrac{1}{3} m^3 (\sec^4 \tfrac{1}{2}\beta - \sec^4 \tfrac{1}{2}\alpha),$$

and therefore

$$\bar{y} = \tfrac{1}{3} m \frac{\sec^4 \frac{1}{2}\beta - \sec^4 \frac{1}{2}\alpha}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}$$

$$= \frac{2m}{3} \frac{\frac{1}{2}(\tan^4 \frac{1}{2}\beta - \tan^4 \frac{1}{2}\alpha) + (\tan^2 \frac{1}{2}\beta - \tan^2 \frac{1}{2}\alpha)}{\frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha) + (\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha)}.$$

Let SQ be a radius vector very near to SP; and let pq, $p'q'$, be two circular arcs described about S as a centre, with radii Sp, Sp', very nearly equal to each other. In the integrations which we have executed for the determination of the values of $\bar{x}$ and $\bar{y}$, we have first conceived the indefinitely thin triangle PSQ to be made up of an infinite series of infinitesimal parallelograms $pqp'q'$, and we have then conceived the whole area KSL to be composed of an infinite number of indefinitely thin triangles, such as PSQ: thus the expressions

$$r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

represent the moments of the area $pqp'q'$ about the axes of y and x; the expressions

$$\int_0^r r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

the moments of the area SPQ about the axes of y and x; and the expressions

$$\int_\alpha^\beta \int_0^r r^2 \cos(\pi - \theta)\, d\theta\, dr, \quad \int_\alpha^\beta \int_0^r r^2 \sin(\pi - \theta)\, d\theta\, dr,$$

the moments of the whole area KSL about the axes of y and x.

Also the expressions

$$r\,d\theta\,dr,\quad \int_0^r r\,d\theta\,dr,\quad \int_\alpha^\beta\int_0^r r\,d\theta\,dr,$$

denote respectively the areas $pqp'q'$, PSQ, KSL.

(4) To find the centre of gravity of the area of a quadrant of a circle.

The equation to the circle being

$$x^2+y^2=a^2,$$

we shall have $$\bar{x}=\frac{4a}{3\pi},\quad \bar{y}=\frac{4a}{3\pi}.$$

(5) AB (fig. 7) is a parabola, of which the equation is $a^{m-1}y=x^m$; to find the centre of gravity of the area $PMNQ$, comprised between two ordinates.

If $AM=\alpha$, $AN=\alpha'$, we shall have

$$\bar{x}=\frac{m+1}{m+2}\,\frac{\alpha'^{m+2}-\alpha^{m+2}}{\alpha'^{m+1}-\alpha^{m+1}},\quad \bar{y}=\frac{m+1}{2(2m+1)a^{m-1}}\,\frac{\alpha'^{2m+1}-\alpha^{2m+1}}{\alpha'^{m+1}-\alpha^{m+1}}.$$

Carré; *Mesure des Surfaces*, &c. p. 80.

(6) Cx, Cy, are asymptotes to an hyperbola EAF, (fig. 8); PM, QN, are parallel to yC; to find the centre of gravity of the area $PMNQ$.

If a, b, be the semiaxes of the hyperbola; Cx, Cy, be taken as the axes of x, y; and CM, CN, be denoted by α, α'; then

$$\bar{x}=\frac{\alpha'-\alpha}{\log\alpha'-\log\alpha},\quad \bar{y}=\frac{1}{8}\,\frac{a^2+b^2}{\alpha\alpha'}\,\frac{\alpha'-\alpha}{\log\alpha'-\log\alpha}.$$

(7) To find the centre of gravity of the portion of the area of the curve $y=\sin x$, between $x=0$ and $x=\pi$.

$$\bar{x}=\tfrac{1}{2}\pi,\ \bar{y}=\tfrac{1}{8}\pi.$$

(8) To find the centre of gravity of the area included between the axes of co-ordinates and the parabola of which the equation is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$$

$$\bar{x}=\tfrac{1}{5}a,\quad \bar{y}=\tfrac{1}{5}b.$$

(9) To find the centre of gravity of the area intercepted between a straight line $y = \beta x$ and a parabola $y^2 = 4mx$.

$$\bar{x} = \frac{8m}{5\beta^2}, \quad \bar{y} = \frac{2m}{\beta}.$$

Sect. 3. *Solid of Revolution.*

Let a solid of revolution be generated by the rotation of a plane curve about the axis of x; then the centre of gravity will be within the axis of x, its position being given by the formula

$$\bar{x} = \frac{\iint xy\,dx\,dy}{\iint y\,dx\,dy} = \frac{\int x\,(y^2 - y'^2)\,dx}{\int (y^2 - y'^2)\,dx},$$

y, y', being the limiting values of y for any assignable value of x; if $y' = 0$, we have

$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx}.$$

If polar co-ordinates be adopted, which are frequently convenient, the formula will be

$$\bar{x} = \frac{\iint r^3 \sin\theta \cos\theta\,d\theta\,dr}{\iint r^2 \sin\theta\,d\theta\,dr},$$

the pole being taken at the origin of x, and θ being the angle of inclination of the radius vector r to the axis of x.

(1) To find the centre of gravity of the segment of a sphere.

The centre of the generating circle being taken as the origin, its equation will be

$$x^2 + y^2 = a^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1);$$

and, c being the distance of the centre of the plane face of the segment from the origin,

$$\bar{x} = \frac{\int_c^a xy^2\,dx}{\int_c^a y^2\,dx} \text{ .... } (2);$$

but
$$\int_c^a xy^2\,dx = \int_c^a (a^2 - x^2)\,x\,dx, \text{ from } (1),$$
$$= \tfrac{1}{2}a^2x^2 - \tfrac{1}{4}x^4, \text{ from } x = c \text{ to } x = a,$$
$$= \tfrac{1}{4}a^4 - \tfrac{1}{2}a^2c^2 + \tfrac{1}{4}c^4 = \tfrac{1}{4}(a^2 - c^2)^2;$$

also
$$\int_c^a y^2\,dx = \int_c^a (a^2 - x^2)\,dx$$
$$= a^2x - \tfrac{1}{3}x^3, \text{ from } x = c \text{ to } x = a,$$
$$= \tfrac{2}{3}a^3 - a^2c + \tfrac{1}{3}c^3:$$

hence from (2),
$$\bar{x} = \tfrac{3}{4}\,\frac{(a^2 - c^2)^2}{2a^3 - 3a^2c + c^3} = \tfrac{3}{4}\,\frac{(a + c)^2}{2a + c}.$$

If the segment become a semi-circle, then $c = 0$, and therefore
$$\bar{x} = \tfrac{3}{8}a.$$

Lucas Valerius; *De Centro Gravitatis Solidorum*, Lib. II. Prop. 33, and Lib. III. Prop. 31. Guldin; *Centrobaryca*, Lib. I. cap. 11, p. 130. Wallis; *Opera*, tom. I. p. 728.

(2) To find the centre of gravity of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

Let β denote the angle between the extreme radii of the sector; then, the centre of the circle being the origin of x, and a the radius,

$$\bar{x} = \frac{\int_0^\beta\int_0^a r^3 \sin\theta\cos\theta\,d\theta\,dr}{\int_0^\beta\int_0^a r^2\sin\theta\,d\theta\,dr} \text{ .... } (1);$$

but
$$\int_0^\beta\int_0^a r^3\sin\theta\cos\theta\,d\theta\,dr = \tfrac{1}{4}a^4\int_0^\beta \sin\theta\cos\theta\,d\theta$$
$$= \tfrac{1}{8}a^4\int_0^\beta \sin 2\theta\,d\theta = \tfrac{1}{16}a^4(1 - \cos 2\beta),$$

and $\displaystyle\int_0^\beta\int_0^a r^2 \sin\theta\, d\theta\, dr = \tfrac{1}{3}a^3\int_0^\beta \sin\theta\, d\theta = \tfrac{1}{3}a^3(1-\cos\beta)$;

hence from (1) we have

$$\bar{x} = \tfrac{3}{16}a\,\frac{1-\cos 2\beta}{1-\cos\beta} = \tfrac{3}{8}a\,(1+\cos\beta) = \tfrac{3}{4}a\cos^2\tfrac{1}{2}\beta.$$

We might equally well have integrated the numerator and denominator of (1), first with respect to θ, and afterwards with respect to r. In the one order of integration, we conceive the sector to be made up of an infinite number of thin triangles, of which the centre of the circle is the common vertex; in the other order, the sector is conceived to be made up of an infinite number of infinitesimal rings, having the centre of the circle as their common centre.

Wallis; *Opera*, Tom. I. p. 728.

(3) To find the centre of gravity of the solid generated by the revolution of the parabolic area ABC (fig. 9), about the tangent Ax at the vertex A, BC being at right angles to the axis Ay of the parabola.

Taking Ax, Ay, as the axes of x, y, the equation to the curve will be

$$x^2 = 4my.$$

Let $AC = a$, $BC = b$; then

$$\bar{x} = \frac{\displaystyle\int_0^b\int_y^a xy\,dx\,dy}{\displaystyle\int_0^b\int_y^a y\,dx\,dy} = \frac{\displaystyle\int_0^b (a^2-y^2)\,x\,dx}{\displaystyle\int_0^b (a^2-y^2)\,dx}$$

$$= 2m^{\frac12}\,\frac{\displaystyle\int_0^a (a^2-y^2)\,dy}{\displaystyle\int_0^a (a^2y^{-\frac12}-y^{\frac32})\,dy}, \text{ from (2)},$$

$$= 2m^{\frac12}\,\frac{a^3-\frac13 a^3}{2a^{\frac52}-\frac25 a^{\frac52}} = \tfrac{5}{6}m^{\frac12}a^{\frac12} = \tfrac{5}{12}b.$$

This is a case of a more general problem given by Carré, *Mesure des Surfaces*, &c. p. 93.

(4) To find the centre of gravity of the solid formed by the revolution of any parabola, of which the equation is

$$y^{m+n} = a^m x^n.$$

For any portion of the solid from $x = 0$ to $x = b$,

$$\bar{x} = \frac{m + 3n}{2m + 4n} b.$$

(5) To find the centre of gravity of the solid generated by the revolution about the axis of x of the curve corresponding to the equation

$$y = (a - x)\left(\frac{x}{a}\right)^2,$$

between the limits $x = 0$ and $x = a$.

$$\bar{x} = \tfrac{4}{9} a.$$

Carré; *Mesure des Surfaces*, &c. p. 99.

(6) To find the centre of gravity of the frustum of a paraboloid.

If a, b, be the radii of the less and of the greater ends, h the length of the frustum, and $\bar{x}$ the distance of the centre of gravity from the smaller end;

$$\bar{x} = \tfrac{1}{3} h \frac{a^2 + 2b^2}{a^2 + b^2}.$$

(7) To find the centre of gravity of an hyperboloid.

If the equation to the generating hyperbola be

$$y^2 = \frac{b^2}{a^2}(x^2 + 2ax),$$

we shall have for the volume between $x = 0$ and $x = c$,

$$\bar{x} = \frac{8ac + 3c^2}{4(3a + c)}.$$

Carré; *Ib.* p. 97.

(8) ABC (fig. 10) is a portion of the area of a common parabola, where BC is at right angles to the axis Ax of the

parabola; to find the centre of gravity of the solid generated by the revolution of the area ABC about BC.

Let $BC=b$; then, G being the centre of gravity,

$$CG=\tfrac{5}{16}b.$$

Carré; *Ib.* p. 90.

(9) AC, BC, (fig. 11) are the semiaxes of an hyperbola, AD being a portion of the curve intercepted by BD drawn parallel to CA; to find the centre of gravity of the solid generated by the revolution of the area $ACBD$ about CB.

If $BC=b$, then, G being the position of the centre of gravity in BC,

$$CG=\tfrac{9}{16}b.$$

Carré; *Ib.* p. 97.

(10) To find the centre of gravity of the solid formed by scooping out a cone from a given paraboloid of revolution, the bases of the two volumes being coincident as well as their vertices.

The centre of gravity bisects the axis.

(11) To find the position of the centre of gravity of the volume included between the surfaces generated by the revolution of two parabolas, $y^2=lx$, $y^2=l'(a-x)$, round the axis of x.

$$\bar{x}=\tfrac{1}{3}a\frac{l+2l'}{l+l'}.$$

Sect. 4. *Any Solid.*

Let x, y, z, be the co-ordinates of any point whatever within any assigned solid; let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity of this solid; then

$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{y}=\frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

where each of the triple integrations is to be performed in accordance with the nature of the bounding surface of the solid.

(1) To find the centre of gravity of a portion of the cone, of which the equation is

$$y^2 + z^2 = \beta^2 x^2,$$

which is contained between the planes of zx, xy, and a given plane parallel to that of yz.

Let a be the length of the axis of the portion of the cone: then

$$\bar{x} = \frac{\int_0^a \int_0^{\beta x} \int_0^z x\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz}, \quad \bar{y} = \frac{\int_0^a \int_0^{\beta x} \int_0^z y\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz},$$

$$\bar{z} = \frac{\int_0^a \int_0^{\beta x} \int_0^z z\,dx\,dy\,dz}{\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz}.$$

Now $$\int_0^a \int_0^{\beta x} \int_0^z dx\,dy\,dz = \int_0^a \int_0^{\beta x} z\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} (\beta^2 x^2 - y^2)^{\frac{1}{2}}\,dx\,dy$$

$$= \int_0^a \tfrac{1}{4}\pi\beta^2 x^2\,dx = \tfrac{1}{12}\pi\beta^2 a^3.$$

Also $$\int_0^a \int_0^{\beta x} \int_0^z x\,dx\,dy\,dz = \int_0^a \int_0^{\beta x} xz\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} x(\beta^2 x^2 - y^2)^{\frac{1}{2}}\,dx\,dy = \int_0^a x\,.\,\tfrac{1}{4}\pi\beta^2 x^2\,dx$$

$$= \tfrac{1}{16}\pi\beta^2 a^4;$$

hence $$\bar{x} = \frac{\tfrac{1}{16}\pi\beta^2 a^4}{\tfrac{1}{12}\pi\beta^2 a^3} = \tfrac{3}{4}a.$$

Again, $$\int_0^a \int_0^{\beta x} \int_0^z y\,dx\,dy\,dz = \int_0^a \int_0^{\beta x} yz\,dx\,dy$$

$$= \int_0^a \int_0^{\beta x} (\beta^2 x^2 - y^2)^{\frac{1}{2}}\,y\,dx\,dy$$

$$= \int_0^a \tfrac{1}{3}\beta^3 x^3\,dx = \tfrac{1}{12}\beta^3 a^4,$$

and therefore $$\bar{y} = \frac{\frac{1}{12}\beta^4 a^4}{\frac{1}{12}\pi\beta^3 a^3} = \frac{\beta}{\pi} a.$$

Similarly, $$\bar{z} = \frac{\beta}{\pi} a.$$

(2) To find the centre of gravity of half the solid intercepted between the surfaces of a hemisphere and a paraboloid of revolution on the same base, the latus-rectum of the paraboloid coinciding with the diameter of the hemisphere, and the solid being bisected by a plane passing through its axis.

Take the centre of the sphere as the origin of co-ordinates, and the axis of the paraboloid as the axis of z; also let the axis of x be so taken that the plane of zx coincides with the bisecting plane; and take the axis of y at right angles to this plane. Then, if a be the radius of the sphere, the equation to the sphere will be

$$x^2 + y^2 + z^2 = a^2,$$

and to the paraboloid,

$$x^2 + y^2 = a(a - 2z).$$

The centre of gravity will be somewhere in the plane of yz, and is to be determined by the formulæ

$$\bar{y} = \frac{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} y\,dx\,dy\,dz}{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz}, \quad \bar{z} = \frac{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} z\,dx\,dy\,dz}{\int_{-a}^{+a}\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz};$$

where x' is taken to represent $(a^2 - x^2)^{\frac{1}{2}}$, and where the limits z', z, of the general value of z, are its values for any assigned values of x and y in the paraboloid and sphere respectively.

Now

$$\int_{z'}^{z} dx\,dy\,dz = dx\,dy\,(z - z') = dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\};$$

hence $$\int_0^{x'}\int_{z'}^{z} dx\,dy\,dz = \int_0^{x'} dx\,dy\,\{(a^2 - x^2 - y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2 - x^2 - y^2)\}$$

$$= dx\,\{\tfrac{1}{4}\pi(a^2 - x^2) - \frac{1}{3a}(a^2 - x^2)^{\frac{3}{2}}\}.$$

and therefore

$$\int_{-a}^{+a}\int_{0}^{x'}\int_{z'}^{z} dx\,dy\,dz = \int_{-a}^{+a} dx\,\{\tfrac{1}{4}\pi(a^2-x^2) - \frac{1}{3a}(a^2-x^2)^{\frac{3}{2}}\}$$

$$= \tfrac{1}{3}\pi a^3 - \frac{1}{3a}\int_{-a}^{+a}(a^2-x^2)^{\frac{3}{2}}\,dx = \tfrac{1}{3}\pi a^3 - \tfrac{1}{8}\pi a^3 = \tfrac{5}{24}\pi a^3.$$

Again, $\int_{z'}^{z} y\,dx\,dy\,dz = y\,dx\,dy\,\{(a^2-x^2-y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2-x^2-y^2)\}$,

$$\int_{0}^{x'}\int_{z'}^{z} y\,dx\,dy\,dz = \int_{0}^{x'} y\,dx\,dy\,\{(a^2-x^2-y^2)^{\frac{1}{2}} - \frac{1}{2a}(a^2-x^2-y^2)\}$$

$= dx\{-\tfrac{1}{3}(a^2-x^2-y^2)^{\frac{3}{2}} - \frac{1}{4a}(a^2-x^2)\,y^2 + \frac{1}{8a}y^4\}$, between limits,

$$= dx\,\{\tfrac{1}{3}(a^2-x^2)^{\frac{3}{2}} - \frac{1}{8a}(a^2-x^2)^2\};$$

hence

$$\int_{-a}^{+a}\int_{0}^{x'}\int_{z'}^{z} y\,dx\,dy\,dz = \tfrac{1}{3}\int_{-a}^{+a}(a^2-x^2)^{\frac{3}{2}}\,dx - \frac{1}{8a}\int_{-a}^{+a}(a^4-2a^2x^2+x^4)\,dx$$

$$= \tfrac{1}{8}\pi a^4 - \tfrac{2}{15}a^4 = \frac{15\pi-16}{120}a^4.$$

Again

$$\int_{z'}^{z} z\,dx\,dy\,dz = \tfrac{1}{2}dx\,dy\,(z^2-z'^2) = \tfrac{1}{2}dx\,dy\,\{(a^2-x^2-y^2) - \frac{1}{4a^2}(a^2-x^2-y^2)^2\},$$

$$\int_{0}^{x'}\int_{z'}^{z} z\,dx\,dy\,dz$$

$$= \tfrac{1}{2}\int_{0}^{x'} dx\,dy\,\{a^2-x^2-y^2 - \frac{1}{4a^2}(a^2-x^2)^2 + \frac{1}{2a^2}(a^2-x^2)\,y^2 - \frac{1}{4a^2}y^4\}$$

$$= \tfrac{1}{2}dx\{(a^2-x^2)^{\frac{3}{2}} - \tfrac{1}{3}(a^2-x^2)^{\frac{3}{2}} - \frac{1}{4a^2}(a^2-x^2)^{\frac{5}{2}} + \frac{1}{6a^2}(a^2-x^2)^{\frac{5}{2}} - \frac{1}{20a^2}(a^2-x^2)^{\frac{5}{2}}\}$$

between limits,

$$= \tfrac{1}{3}dx\,\{(a^2-x^2)^{\frac{3}{2}} - \frac{1}{5a^2}(a^2-x^2)^{\frac{5}{2}}\},$$

$$\int_{-a}^{+a}\int_{0}^{x'}\int_{z'}^{z} z\,dx\,dy\,dz = \tfrac{1}{3}\int_{-a}^{+a} dx\,\{(a^2-x^2)^{\frac{3}{2}} - \frac{1}{5a^2}(a^2-x^2)^{\frac{5}{2}}\}$$

$$= \tfrac{1}{3}\int_{-a}^{+a} dx\,(a^2-x^2)^{\frac{3}{2}} - \frac{1}{15a^2}\int_{-a}^{+a}(a^2-x^2)^{\frac{5}{2}}\,dx$$

$$= \tfrac{1}{8}\pi a^4 - \tfrac{1}{48}\pi a^4 = \tfrac{5}{48}\pi a^4.$$

From the formulæ for $\bar{y}$ and $\bar{z}$ then we have

$$\bar{y} = \frac{\frac{1}{120}(15\pi - 16)a^4}{\frac{5}{24}\pi a^3} = \frac{15\pi - 16}{25\pi}a,$$

$$\bar{z} = \frac{\frac{5}{48}\pi a^4}{\frac{5}{24}\pi a^3} = \frac{1}{2}a.$$

(3) AOC (fig. 12) is a right-angled triangle, O being the right angle; $AOBD$ is a rectangle, of which the plane is perpendicular to that of the triangle; from every point R in the line AC a straight line RQ is drawn to meet BD in Q, in a plane at right angles to the areas of the rectangle and triangle; to find the centre of gravity of the volume so generated.

Let OAx, OBy, OCz, be taken as axes of x, y, z; from R draw RM at right angles to OA, join QM, and draw PN at right angles to QM; let $OA = a$, $OB = b$, $OC = c$; $OM = x$, $MN = y$, $NP = z'$, z being the distance of any point in the line PN from the point N: then for the determination of the centre of gravity we have

$$\bar{x} = \frac{\int_0^a\int_0^b\int_0^{z'} x\,dx\,dy\,dz}{\int_0^a\int_0^b\int_0^{z'} dx\,dy\,dz}, \quad \bar{y} = \frac{\int_0^a\int_0^b\int_0^{z'} y\,dx\,dy\,dz}{\int_0^a\int_0^b\int_0^{z'} dx\,dy\,dz},$$

$$\bar{z} = \frac{\int_0^a\int_0^b\int_0^{z'} z\,dx\,dy\,dz}{\int_0^a\int_0^b\int_0^{z'} dx\,dy\,dz}.$$

From the geometry it is evident that

$$z' = c\frac{a-x}{a}\,\frac{b-y}{b};$$

hence we have

$$\bar{x} = \frac{\int_0^a\int_0^b x(a-x)(b-y)\,dx\,dy}{\int_0^a\int_0^b (a-x)(b-y)\,dx\,dy}$$

$$=\frac{\int_0^a x(a-x)\,dx}{\int_0^a (a-x)\,dx}=\frac{\frac{1}{6}a^3}{\frac{1}{2}a^2}=\tfrac{1}{3}a.$$

$$\bar{y}=\frac{\int_0^a\int_0^b (a-x)(b-y)\,y\,dx\,dy}{\int_0^a\int_0^b (a-x)(b-y)\,dx\,dy}=\frac{\int_0^b (b-y)\,y\,dy}{\int_0^b (b-y)\,dy}=\frac{\frac{1}{6}b^3}{\frac{1}{2}b^2}=\tfrac{1}{3}b,$$

$$\bar{z}=\frac{\int_0^a\int_0^b \frac{1}{2}c^2\frac{(a-x)^2}{a^2}\frac{(b-y)^2}{b^2}\,dx\,dy}{\int_0^a\int_0^b c\,\frac{a-x}{a}\cdot\frac{b-y}{b}\,dx\,dy}=\frac{c}{2ab}\,\frac{\int_0^a\int_0^b (a-x)^2(b-y)^2\,dx\,dy}{\int_0^a\int_0^b (a-x)(b-y)\,dx\,dy}$$

$$=\frac{c}{2ab}\,\frac{\frac{1}{3}a^3\cdot\frac{1}{3}b^3}{\frac{1}{2}a^2\cdot\frac{1}{2}b^2}=\frac{2c}{9}.$$

(4) To find the centre of gravity of the portion of the sphere

$$x^2+y^2+z^2=a^2,$$

which is cut off by three planes, $x=0$, $y=0$, $z=0$.

$$\bar{x}=\bar{y}=\bar{z}=\tfrac{3}{8}a.$$

(5) To find the centre of gravity of a portion of the paraboloid

$$y^2+z^2=4mx,$$

which is cut off by the three planes $x=a$, $y=0$, $z=0$.

If b be the radius of the section of the paraboloid made by the plane $x=a$, then

$$\bar{x}=\tfrac{2}{3}a,\quad \bar{y}=\bar{z}=\frac{16b}{15\pi}.$$

(6) To find the centre of gravity of a portion of the solid $z^2=xy$, which is cut off by the five planes $x=0$, $y=0$, $z=0$, $x=a$, $y=b$.

$$\bar{x}=\tfrac{3}{5}a,\quad \bar{y}=\tfrac{3}{5}b,\quad \bar{z}=\tfrac{9}{32}a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

(7) To find the centre of gravity of the volume of the cylinder $y^2=2ax-x^2$, which is cut off between the two planes $z=\beta x$, $z=\beta' x$.

$$\bar{x}=\tfrac{5}{4}a,\quad \bar{y}=0,\quad \bar{z}=\tfrac{5}{8}(\beta+\beta')\,a.$$

(8) A solid is generated by a variable rectangle moving parallel to itself along an axis perpendicular to its plane through its centre; one side of the rectangle varies as the distance from a fixed point in the axis, while half the other is the sine of a circular arc, of which this distance is the versed sine; to determine the distance of the centre of gravity of the whole solid from the fixed point.

The required distance is equal to five-eighths of the length of the axis.

SECT. 5. *A Plane Curve.*

Let x, y, be the co-ordinates of any point of a plane curve, and let ds denote an element of the length of the curve at that point; then, $\bar{x}$, $\bar{y}$, denoting the co-ordinates of the centre of gravity of any assigned portion of the curve,

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds},$$

the integrations being performed in accordance with the limits of the portion.

The idea of the determination of the centres of gravity of curve lines is due to La-Faille, a Flemish mathematician, by whom it was applied in the instances of portions of the circle and the ellipse, in a work entitled "*De centro gravitatis partium circuli et ellipsis theoremata,*" published in the year 1632. The theorems of La-Faille were afterwards published in a somewhat more elegant form, and with amplifications, by Guldin; *Centrobaryca,* Lib. I. caps. 4, 5, 6, 7.

(1) To find the centre of gravity of the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

From the equation to the curve we have

$$ds^2 = dx^2 + dy^2 = (1 + \cos^2 x)\,dx^2;$$

hence $$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^\pi \sin x\,(1+\cos^2 x)^{\frac{1}{2}}\,dx}{\int_0^\pi (1+\cos^2 x)^{\frac{1}{2}}\,dx}.$$

Now, by the ordinary processes of the integral calculus,

$$\int_0^\pi \sin x\,(1+\cos^2 x)^{\frac{1}{2}}\,dx = 2^{\frac{1}{2}} + \log(2^{\frac{1}{2}}+1);$$

also, c denoting the length of the curve from $x=0$ to $x=\pi$,

$$c = \int_0^\pi (1+\cos^2 x)^{\frac{1}{2}}\,dx = 2^{\frac{1}{2}}\int_0^\pi (1-\tfrac{1}{2}\sin^2 x)^{\frac{1}{2}}\,dx,$$

an elliptic function of the second order: hence $\bar{y}$ is given by the equation

$$c\bar{y} = 2^{\frac{1}{2}} + \log(2^{\frac{1}{2}}+1).$$

(2) To find the centre of gravity of any arc of a circle.

Let the centre of the circle be taken as the origin of co-ordinates, and let the axis of x bisect the arc; then, if a be the radius of the circle, c the chord of the arc, and s the length of the arc,

$$\bar{x} = \frac{ac}{s}, \quad \bar{y} = 0.$$

Guldin; *Centrobaryca*, Lib. I. cap. 5, p. 59.
Wallis; *Opera*, Tom. I. p. 712.

(3) To find the centre of gravity of the arc of a semicycloid.

The equations to the curve being

$$x = a(1-\cos\theta), \quad y = a(\theta+\sin\theta),$$

we shall have

$$\bar{x} = \tfrac{4}{3}a, \quad \bar{y} = (\pi - \tfrac{4}{3})\,a.$$

Wallis; *Opera*, Tom. I. p. 520.

(4) To find the centre of gravity of the arc of a catenary

$$y = \tfrac{1}{2}a\,(e^{\frac{x}{a}} + e^{-\frac{x}{a}}),$$

cut off by any assigned double ordinate.

If $2s$ be the whole length of the intercepted arc,

$$\bar{x}=0, \quad \bar{y}=\frac{ax+sy}{2s}.$$

(5) To find the centre of gravity of the arc of a parabola $y^2 = 4mx$, cut off by the latus rectum.

$$\bar{x}=\tfrac{1}{4}m\,\frac{3.2^{\frac{1}{2}}-\log(1+2^{\frac{1}{2}})}{2^{\frac{1}{2}}+\log(1+2^{\frac{1}{2}})}, \quad \bar{y}=0.$$

(6) To find the centre of gravity of the semi-arc of a loop of the Lemniscata of James Bernoulli.

If the axis of the loop be taken as the axis of x, the node being the origin; then, a being the length of the axis and l of the semi-arc,

$$\bar{x}=\frac{a^2}{2^{\frac{1}{2}}l}, \quad \bar{y}=\frac{(2^{\frac{1}{2}}-1)\,a^2}{2^{\frac{1}{2}}l}.$$

SECT. 6. *Curve of Double Curvature.*

The formulæ for the determination of the centre of gravity of a curve of double curvature, are

$$\bar{x}=\frac{\int x\,ds}{\int ds}, \quad \bar{y}=\frac{\int y\,ds}{\int ds}, \quad \bar{z}=\frac{\int z\,ds}{\int ds};$$

where x, y, z, are the co-ordinates of any point in the curve, and ds an element of the arc at that point: the limits of the integrations will depend upon the positions of the ends of that portion of the curve of which the centre of gravity is required.

Ex. To find the centre of gravity of the Helix.

The equations to the curve are

$$x^2+y^2=a^2, \qquad z=b\cos^{-1}\frac{x}{a};$$

and for the centre of gravity of any length of the curve, beginning at the origin of co-ordinates,

$$\bar{x} = \frac{by}{z}, \quad \bar{y} = \frac{b(a-x)}{z}, \quad \bar{z} = \tfrac{1}{2}z.$$

SECT. 7. *Surface of Revolution.*

Let x, y, be the co-ordinates of any point of a curve, by the revolution of which about the axis of x a surface is supposed to be generated; then, if ds denote an element of the generating curve at the point, we have for the position of the centre of gravity of the surface of revolution in the axis of x,

$$\bar{x} = \frac{\int xy\, ds}{\int y\, ds};$$

the integrations being performed between limits depending upon the magnitude of the surface.

(1) To find the centre of gravity of the surface of a segment of a sphere.

If the equation to the generating circle be

$$y = (2ax - x^2)^{\frac{1}{2}},$$

we shall have
$$dy = \frac{a - x}{(2ax - x^2)^{\frac{1}{2}}}\, dx,$$

and therefore

$$ds^2 = dx^2 + dy^2 = \frac{a^2}{2ax - x^2}\, dx^2 = \frac{a^2 dx^2}{y^2}, \quad \text{or } y\, ds = a\, dx;$$

hence for any segment, of which the limiting abscissa is c,

$$\bar{x} = \frac{\int_0^c ax\, dx}{\int_0^c a\, dx} = \frac{\frac{1}{2}c^2}{c} = \tfrac{1}{2}c.$$

(2) To find the centre of gravity of the surface generated by the revolution of one of the loops of the curve $r^2 = a^2 \cos 2\theta$ about its axis.

$$\bar{x} = \frac{\int xy\, ds}{\int y\, ds} = \frac{\int_0^{\frac{1}{4}\pi} r^2 \sin\theta \cos\theta\, (dr^2 + r^2 d\theta^2)^{\frac{1}{2}}}{\int_0^{\frac{1}{4}\pi} r \sin\theta\, (dr^2 + r^2 d\theta^2)^{\frac{1}{2}}}.$$

But $\qquad r = a (\cos 2\theta)^{\frac{1}{2}}, \qquad dr = -a \dfrac{\sin 2\theta}{(\cos 2\theta)^{\frac{1}{2}}} d\theta,$

and therefore $\qquad dr^2 + r^2\, d\theta^2 = \dfrac{a^2\, d\theta^2}{\cos 2\theta}.$

Hence
$$\bar{x} = a \,.\, \frac{\int_0^{\frac{1}{4}\pi} \sin\theta \cos\theta\, (\cos 2\theta)^{\frac{1}{2}}\, d\theta}{\int_0^{\frac{1}{4}\pi} \sin\theta\, d\theta}$$

$$= \tfrac{1}{4} a \frac{\int (\cos 2\theta)^{\frac{1}{2}}\, d\cos 2\theta}{\int d\cos\theta}$$

$$= \tfrac{1}{4} a \frac{\left\{\frac{2}{3} (\cos 2\theta)^{\frac{3}{2}}\right\}_0^{\frac{1}{4}\pi}}{\left\{\cos\theta\right\}_0^{\frac{1}{4}\pi}} = \tfrac{1}{4} a \frac{-\frac{2}{3}}{\frac{1}{\sqrt{2}} - 1} = \frac{a}{6} \cdot \frac{\sqrt{2}}{\sqrt{(2)} - 1}.$$

(3) To find the centre of gravity of the surface of a cone.

Let the equation to the generating straight line be $y = ax$; then, c being the length of the axis of the cone,

$$\bar{x} = \tfrac{2}{3} c.$$

Guldin; *Centrobaryca*, Lib. I. cap. 10, prop. 3.

(4) To find the centre of gravity of the surface generated by the revolution of a semicycloid about its axis.

The equations to the curve being

$$x = a (1 - \cos\theta), \qquad y = a (\theta + \sin\theta),$$

we shall have

$$\bar{x} = \tfrac{2}{15} a \, \frac{15\pi - 8}{3\pi - 4}.$$

(5) To find the centre of gravity of the surface generated by the revolution of the parabola $y^2 = 4mx$ about the axis of x.

$$\bar{x} = \frac{(3x - 2m)(x + m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}}.$$

Sect. 8. *Any Surface.*

Let x, y, z, be the co-ordinates of any point of a surface referred to three rectangular axes; and let $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$; then, for the centre of gravity of any portion of the surface,

$$\bar{x} = \frac{\iint x\,(1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy},$$

$$\bar{y} = \frac{\iint y\,(1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy},$$

$$\bar{z} = \frac{\iint z\,(1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}}\,dx\,dy},$$

the integrations being performed between limits corresponding to the boundary of the surface.

(1) Suppose the surface to be any portion of the superficies of a sphere, of which the equation is

$$x^2 + y^2 + z^2 = r^2.$$

Then we have

$$p = -\frac{x}{z}, \qquad q = -\frac{y}{z};$$

and therefore $$\bar{z} = r \frac{\iint dx\, dy}{\iint dx\, dy\, (1 + p^2 + q^2)^{\frac{1}{2}}}.$$

Now it is evident that, the integrations being performed within the given limits, the denominator of this expression for $\bar{z}$ represents the area of the given portion of the surface, while the numerator represents the area of the projection of this same portion upon the plane of x and y. Hence in general language: the distance of the centre of gravity of any portion whatever of the surface of a sphere from the plane of any one of its great circles, is a fourth proportional to the area of the portion itself, the area of its projection on this plane, and the radius of the sphere.

The truth of this proposition depends solely upon the property expressed by the equation

$$z\,(1 + p^2 + q^2)^{\frac{1}{2}} = r\,;$$

but this equation holds good for the whole class of surfaces generated by the motion of a sphere of invariable radius, of which the centre describes a plane curve traced arbitrarily in the plane of x and y; hence we may evidently extend the preceding proposition to all these surfaces under the following enunciation:—

"Upon any surface whatever, generated by the motion of a sphere of which the centre never departs from a given plane, let any portion S be taken, and let S' be the projection of S upon the given plane; then the distance of the centre of gravity of S from this plane will be a fourth proportional to S, S', and the radius of the generating sphere."

(2) To find the centre of gravity of any spherical triangle formed by three great circles.

Let ABC (fig. 13) be any spherical triangle, O the centre of the sphere; and OA, OB, OC, the three radii at the angles A, B, C, of the triangle. Let Z_a, Z_b, Z_c, denote the distances of the centre of gravity of this triangle from the three planes

BOC, COA, AOB; then, by the proposition of the preceding Article, if r be the radius of the sphere, S the area of the spherical triangle ABC, and S' of its projection upon the plane BOC,

$$Z_a = \frac{S'}{S} r.$$

But, by the principles of spherical trigonometry,

$$S = \frac{\pi r^2}{180}(A + B + C - 180);$$

also it is clear that, a, b, c, being the number of degrees subtended at the centre of the sphere by the sides of the spherical triangle opposite to the angles A, B, C,

$$S' = \text{area } BOC - \text{area } AOB \times \cos B - \text{area } AOC \times \cos C$$

$$= \frac{\pi r^2}{360}(a - c \cos B - b \cos C),$$

and therefore $$Z_a = \tfrac{1}{2} r \frac{a - b \cos C - c \cos B}{A + B + C - 180}.$$

Similarly, $$Z_b = \tfrac{1}{2} r \frac{b - c \cos A - a \cos C}{A + B + C - 180}.$$

$$Z_c = \tfrac{1}{2} r \frac{c - a \cos B - b \cos A}{A + B + C - 180}.$$

The position of the centre of gravity of the spherical triangle may be elegantly expressed likewise in terms of its distances from three great circles of the sphere, at right angles to the three edges OA, OB, OC, of the spherical pyramid $ABCO$. Let D_a, D_b, D_c, denote these distances; then by Art. (1) we have

$$D_a = r \frac{S_a}{S}, \quad D_b = r \frac{S_b}{S}, \quad D_c = r \frac{S_c}{S},$$

where S denotes the spherical area ABC, and S_a, S_b, S_c, its projections upon the three great circles at right angles to OA, OB, OC.

Now it is evident that the projections of the spherical area ABC, and of the sector BOC, upon the great circle which is at

right angles to OA, are identically the same, and therefore, if the arc $A\alpha$ be drawn at right angles to BC, we have

$$S_a = \text{area of sector } BOC \times \cos\left(\frac{\pi}{2} - \frac{A\alpha}{r}\right)$$

$$= \frac{\pi}{360} ar^2 \sin\frac{A\alpha}{r} = \frac{\pi}{360} ar^2 \sin B . \sin c :$$

but $$S = \frac{\pi r^2}{180}(A + B + C - 180);$$

hence $$D_a = \tfrac{1}{2} r \frac{a \sin B \sin c}{A + B + C - 180}.$$

Similarly, $$D_b = \tfrac{1}{2} r \frac{b \sin C \sin a}{A + B + C - 180},$$

$$D_c = \tfrac{1}{2} r \frac{c \sin A \sin b}{A + B + C - 180}.$$

If we desire to determine the position of the centre of gravity of the triangle by means of three rectangular co-ordinates x, y, z, let the plane of the side c be taken as the plane of x and y, and let the radius OA be taken to coincide with the axis of x. Then from the preceding results we have at once

$$x = \tfrac{1}{2} r \frac{a \sin B \sin c}{A + B + C - 180}, \quad z = \tfrac{1}{2} r \frac{c - b \cos A - a \cos B}{A + B + C - 180}.$$

Again, let the great circle, of which BC is an arc, meet the plane of x, z, in the point D, as in fig. 14; join A and D by an arc of a great circle. Then clearly the projection of the spherical triangle ABC upon the plane of x and z, is equal to the difference of the projections of the sectors AOC, BOC, upon this plane, and therefore to the expression

$$\frac{\pi}{360} r^2 b \cos CAD - \frac{\pi}{360} r^2 a \cos D$$

$$= \frac{\pi r^2}{360}(b \sin A - a \sin B \cos c);$$

hence, by the principles of Art. (1), we have

$$y = \tfrac{1}{2} \frac{b \sin A - a \sin B \cos c}{A + B + C - 180}.$$

(3) The general formula

$$\bar{z} = \frac{\iint z\,(1+p^2+q^2)^{\frac{1}{2}}\,dx\,dy}{\iint (1+p^2+q^2)^{\frac{1}{2}}\,dx\,dy},$$

furnishes us with the following general proposition:—

"Upon the surface (A), generated by the revolution of the curve of equilibrium of a homogeneous catenary about the vertical line which passes through its lowest point, trace arbitrarily a perimeter enclosing a portion S of the surface; project this perimeter upon a horizontal plane which intersects the axis of revolution at a distance c below the lowest point of the surface, where c is equal to the horizontal tension of the catenary divided by the mass of a unit of its length; let V be the volume contained between the surface S, its projection, and the cylindrical surface formed by the perpendiculars from the perimeter of S upon the plane of projection. Then the altitude of the centre of gravity of S above this plane will be double of that of the centre of gravity of V."

In fact, the plane touching the surface (A) in a point situated at an altitude z above the plane of projection, which we shall take for the plane of x and y, makes with this plane an angle, of which the cosine is $\frac{c}{z}$; and therefore we have the equation

$$(1+p^2+q^2)^{\frac{1}{2}} = \frac{z}{c};$$

hence, by the formula for $\bar{z}$, we obtain

$$\bar{z} = \frac{\iint z^2\,dx\,dy}{\iint z\,dx\,dy}.$$

But calling $\underline{z}$ the altitude of the centre of gravity of V above the same plane, we have

$$\underline{z} = \frac{\iint \frac{1}{2}\,zz\,dx\,dy}{\iint z\,dx\,dy},$$

and, the limits of the integrations being the same in the expressions for $\bar{z}$ and $\underline{z}$, we see that $\bar{z} = 2\underline{z}$.

The property expressed by the partial differential equation

$$(1 + p^2 + q^2)^{\frac{1}{2}} = \frac{z}{c},$$

being common to all the surfaces which can be generated by the surface (A) when it moves in such a manner that its axis always remains vertical, and that one of its points describes a plane curve traced arbitrarily upon a horizontal plane, the proposition which we have demonstrated is susceptible of the same extension as that of (1).

The illustrations of the general formulæ for the determination of the centre of gravity of any surface, which we have given in (1), (2), (3), are extracted from a memoir by Professor Giulio, of Turin, which may be seen in Liouville's *Journal de Mathématiques*, Tom. IV. p. 386.

(4) To find the centre of gravity of the surface of a cone

$$y^2 + z^2 = \beta^2 x^2,$$

intercepted by the planes $x = 0$, $y = 0$, $z = 0$, $x = a$.

$$\bar{x} = \tfrac{2}{3}a, \quad \bar{y} = \bar{z} = \tfrac{4}{3}\frac{\beta a}{\pi}.$$

SECT. 9. *Heterogeneous Bodies.*

In the preceding sections we have determined the centres of gravity of various classes of homogeneous bodies; we will now give a few examples of the determination of the centre of gravity when the density is variable.

(1) To find the centre of gravity of the surface of a hemisphere, when the density of each point in the surface varies as its perpendicular distance from the circular base of the hemisphere.

Let the equation to the quadrantal arc, by the revolution of which the hemispherical surface may be generated, be

$$x^2 + y^2 = a^2 \text{..........................} (1),$$

the axis of x being the axis of revolution.

The area of the strip of the surface which is generated by the element ds of the arc, will be equal to $2\pi y\, ds$; and, if ρ be its density, its mass will be equal to $2\pi\rho\, y\, ds$: hence

$$\bar{x} = \frac{\int 2\pi\rho\, y\, ds \,.\, x}{\int 2\pi\rho\, y\, ds} = \frac{\int \rho\, xy\, ds}{\int \rho\, y\, ds};$$

but $\rho \propto x$; hence $\quad \bar{x} = \dfrac{\int x^2 y\, ds}{\int xy\, ds};$

and therefore, since from (1) it is easily seen that

$$y\, ds = a\, dx,$$

we have $\quad \bar{x} = \dfrac{\int_0^a x^2\, dx}{\int_0^a x\, dx} = \dfrac{\frac{1}{3}a^3}{\frac{1}{2}a^2} = \frac{2}{3}a.$

(2) To find the centre of gravity of a physical line, the density of which at any point varies as the n^{th} power of its distance from a given point in the line produced.

Let a, b, be the distances of the given point from the two extremities, and $\bar{x}$ its distance from the centre of gravity of the physical line; then

$$\bar{x} = \frac{n+1}{n+2} \frac{b^{n+2} - a^{n+2}}{b^{n+1} - a^{n+1}}.$$

(3) To find the centre of gravity of a quadrant of a circle, the density at any point of which varies as the n^{th} power of its distance from the centre.

Let a denote the radius of the circle, and $\bar{x}, \bar{y}$, the co-ordinates of the centre of gravity of the quadrant referred to its two extreme radii as axes; then

$$\bar{x} = \frac{n+2}{n+3} \cdot \frac{2a}{\pi} = \bar{y}.$$

(4) To find the centre of gravity of a square *ABCD*, the density of which at any point varies as the distance of the point from a line through *A* parallel to *BD*.

The distance of the centre of gravity from *A* is equal to $\frac{7}{12}a$, a denoting the length of a diagonal.

SECT. 10. *Centre of Parallel Forces.*

When any number of parallel forces act on a system of rigidly connected points, they generally have a single resultant acting on a point of which the position is invariable while their common direction is changed in every possible way. This point is called the Centre of the Parallel Forces: the Centre of Gravity of a body is a particular case of this. Let x, y, z, denote the co-ordinates of the point of application of any force P of the system referred to any axes, rectangular or oblique; and let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the Centre of Parallel Forces. Then, R representing the resultant,

$$R = \Sigma(P), \quad \bar{x} = \frac{\Sigma(Px)}{\Sigma(P)}, \quad \bar{y} = \frac{\Sigma(Py)}{\Sigma(P)}, \quad \bar{z} = \frac{\Sigma(Pz)}{\Sigma(P)}.$$

Whenever $\Sigma(P)$ is equal to zero, these formulæ cease to be applicable, there being in this case no single resultant; the forces will be reducible to a resultant couple. For the complete development of the theory of Statical Couples, the reader is referred to Poinsot's beautiful work entitled *Elémens de Statique.*

The formulæ for $\bar{x}$, $\bar{y}$, $\bar{z}$, were first given by Varignon, in the *Mémoires de l'Académie des Sciences de Paris* for the year 1714.

(1) Three parallel forces acting at the angular points A, B, C, of a plane triangle, are respectively proportional to the opposite sides a, b, c; to find the distance of the centre of parallel forces from A.

Produce AB, AC, indefinitely to points x, y, and let Ax, Ay, be taken as co-ordinate axes.

Let μa, μb, μc, be the forces applied at A, B, C, where a, b, c, denote the opposite sides of the triangle. The co-ordinates of

the points of application of these three forces are $0, 0$; $c, 0$; $0, b$; hence

$$\bar{x} = \frac{c.\mu b}{\mu a + \mu b + \mu c} = \frac{bc}{a+b+c},$$

$$\bar{y} = \frac{b.\mu c}{\mu a + \mu b + \mu c} = \frac{bc}{a+b+c} = \bar{x}.$$

Let r be the distance of the centre of parallel forces from A; then

$$r^2 = \bar{x}^2 + \bar{y}^2 + 2\bar{x}\bar{y}\cos A = 2\bar{x}^2(1+\cos A) = 4\bar{x}^2\cos^2\tfrac{1}{2}A,$$

and therefore

$$r = 2\bar{x}\cos\tfrac{1}{2}A = \frac{2bc\cos\frac{1}{2}A}{a+b+c}.$$

(2) Three parallel forces P, Q, R, act at the angles A, B, C, of a given triangle, and are to each other as the reciprocals of the opposite sides a, b, c; to determine the distance of their centre from A.

$$\text{Required distance} = a\frac{(b^4 + 2b^2c^2\cos A + c^4)^{\frac{1}{2}}}{bc + ca + ab}.$$

(3) At the corners B, C, D, of a quadrilateral pyramid $ABCD$, three parallel forces P, Q, R, are applied; to find the distance of their centre from the corner A.

Let $AB = b$, $AC = c$, $AD = d$; $\angle CAD = (c, d)$, $\angle DAB = (d, b)$, $\angle BAC = (b, c)$; $r =$ the required distance; then

$$r^2(P+Q+R)^2 = P^2b^2 + Q^2c^2 + R^2d^2 + 2QRcd\cos(c, d)$$
$$+ 2RPdb\cos(d, b) + 2PQbc\cos(b, c).$$

(4) At three fixed points (a, b), (a', b'), (a'', b''), in the plane of x, y, are applied three parallel forces p, p', p''; supposing the magnitude of p'' to vary in every possible way, to find the locus of the centre of parallel forces.

The locus will be a straight line of which the equation is

$$(ap + a'p')\,b'' + \{(a''-a)\,p + (a''-a')\,p'\}\,y$$
$$= (bp + b'p')\,a'' + \{(b''-b)\,p + (b''-b')\,p'\}\,x.$$

SECT. 11. *The Properties of Pappus.*

I. If any plane area revolve about any axis in its own plane through any assigned angle, the volume of the surface generated by the motion of the area will be equal to a prism, of which the base is equal to the revolving area, and the altitude to the length of the path described by the centre of gravity of the area during its revolution.

II. If any plane area revolve through any angle about any axis in its own plane, the area of the surface generated by its perimeter will be equal to a rectangle, of which one side is the length of the perimeter, and the other the length of the path described by the centre of gravity of the perimeter.

The enunciation of these properties, which are generally called Guldin's properties, is due to Pappus[1], and may be seen at the end of the Preface to the seventh book of his *Mathematical Collections*, of which the first edition appeared in the form of a Latin translation in the year 1588. They were afterwards published, with various applications, by Guldin, in his treatise *De Centro Gravitatis*, Lib. 2 and 3, which appeared for the first time in the year 1635. Cavalieri[2], in reply to objections advanced by Guldin against his method of indivisibles, gave a demonstration of these properties by this method; stating likewise, in allusion to Guldin's claims as a discoverer, that they had been communicated to him, long before the publication of Guldin's work, by a pupil of his, Antonio Roccha. Elegant demonstrations of these properties were given also by Varignon in the *Mémoires de l'Académie des Sciences de Paris* for the year 1714, p. 77.

(1) From any point P (fig. 15) in a parabola, is drawn a

[1] The words of Pappus in the Latin translation are: "Perfectorum utrorumque ordinum proportio composita est ex proportione amphismatum, et rectarum linearum similiter ad axes ductarum a punctis, quæ in ipsis gravitatis centra sunt. Imperfectorum autem proportio composita est ex proportione amphismatum, et circumferentiarum a punctis quæ in ipsis sunt centra gravitatis, factarum, &c." In the former case he is alluding to those solids which are formed by the entire revolution of the generating figures through 360°; in the latter, to those which are formed by revolution through any smaller angle.

[2] *Exercitationes Geometricæ Sex*, Exercit. 1 & 2; Bononiæ, 1647.

straight line PM at right angles to the axis, and meeting it in the point M; to find the content of the solid generated by the complete revolution of the area APM about PM.

Let $AM = x$, $PM = y$; $V =$ the required volume; and $\bar{x} =$ the distance of the centre of gravity of the area APM from PM. Then the whole path described by the centre of gravity will be equal to $2\pi\bar{x}$; hence, by (I.),

$$V = 2\pi\bar{x} \times \text{area of } PAM:$$

but $\bar{x} = \frac{2}{5}x$, and area of $PAM = \frac{2}{3}xy$; and therefore

$$V = 2\pi \cdot \tfrac{2}{5}x \cdot \tfrac{2}{3}xy = \tfrac{8}{15}\pi x^2 y.$$

Complete the parallelogram $MPmA$; then the area of this parallelogram will be equal to xy, and the distance of its centre of gravity from PM will be equal to $\frac{1}{2}x$. Conceive this parallelogram to make an entire revolution about PM; then the path of its centre of gravity will be equal to

$$2\pi \cdot \tfrac{1}{2}x = \pi x;$$

and therefore, if U denote the volume of the cylinder which is generated by the revolution,

$$U = \pi x \cdot xy = \pi x^2 y.$$

Hence $$U : V :: 15 : 8.$$

This is one of the problems proposed in Kepler's *Stereometria*.

Guldinus; *De Centro Gravitatis*, Lib. II. cap. 12, prop. 6.

(2) To find the surface of a sphere.

Let BAb (fig. 16) be a semicircle, by the revolution of which about the diameter BCb the sphere is generated.

Let CA be at right angles to Bb, C being the centre of the circle, and let G be the centre of gravity of the semicircular arc BAb. Let $CA = a$, surface required $= S$; then, by (II.),

$$2\pi CG \cdot \text{arc } BAb = S;$$

but $CG = \dfrac{2a}{\pi}$, and arc $BAb = \pi a$; hence

$$S = 2\pi \cdot \frac{2a}{\pi} \cdot \pi a = 4\pi a^2.$$

Now πa^2 is the area of a great circle of the sphere; and thus we find that the whole surface of a sphere is four times as great as that of one of its great circles. This proposition was first proved by Archimedes, Περὶ Σφαίρας καὶ Κυλίνδρου, Βιβλ. Α, πρότα Λ; and afterwards, according to the method which we have given, by Guldin, *De Centro Gravitatis*, Lib. IV. cap. 1, prop. 7.

(3) To find the distance of the centre of gravity of the area of a semi-parabola from the axis of the parabola.

Let APM (fig. 15) be the semi-parabola.

Let $AM = x$, $PM = y$, $\bar{y} =$ the distance of the centre of gravity of the area APM from AM. Then, since the area of APM is equal to $\frac{2}{3}xy$, and since the volume generated by the revolution of APM about AM is equal to

$$\int \pi y^2 dx = 4\pi m \int x dx = 2\pi m x^2;$$

we have, by (I.),

$$\frac{2}{3}xy \,.\, 2\pi\bar{y} = 2\pi m x^2,$$

$$\frac{2}{3}xy \,.\, \bar{y} = m x^2,$$

$$\frac{8}{3}m x^2 \bar{y} = m x^2 y,$$

$$\bar{y} = \frac{3}{8}y.$$

(4) To find the volume and the surface of the solid ring generated by the complete revolution of a circle about any external line in its own plane.

Let b be the distance of the centre of the circle from the axis of revolution, and a the radius of the circle; then

$$\text{the volume} = 2\pi^2 a^2 b, \text{ and the surface} = 4\pi^2 ab.$$

(5) To find the volume of the solid ring generated by the revolution of an ellipse about an external axis in its own plane through an angle of 180°.

If a, b, be the semiaxes of the ellipse, and c the distance of its centre from the axis of rotation, then

$$\text{the volume} = \pi^2 abc.$$

(6) To find the volume generated by the revolution through a given angle of a portion APM (fig. 15) of a parabola about a tangent at its vertex A, PM being parallel to the tangent, and AM at right angles to it.

If $AM = x$, $PM = y$, and β be the angle through which the revolution takes place; then

$$\text{the volume} = \tfrac{2}{5}\beta x^2 y.$$

(7) To find the volume and the surface of the solid generated by the complete revolution of a cycloid about its axis.

If a be the radius of the generating circle,

$$\text{the volume} = \pi a^3 \left(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right), \quad \text{the surface} = 8\pi a^2 \left(\pi - \tfrac{4}{3}\right).$$

(8) To find the volume and the surface of the solid generated by the complete revolution of a cycloid about its base.

$$\text{The volume} = 5\pi^2 a^3, \quad \text{the surface} = \tfrac{64}{3}\pi a^2.$$

(9) To find the content of the solid generated by the complete revolution of a right-angled triangle about its hypotenuse.

If a, b, denote the two sides of the triangle, the content will be equal to

$$\frac{\pi a^2 b^2}{3\,(a^2 + b^2)^{\frac{1}{2}}}.$$

CHAPTER II.

EQUILIBRIUM OF A PARTICLE.

LET P denote any one of a system of forces acting on a particle; and let α, β, γ, be the angles which the direction of this force makes with any three proposed straight lines, no two of which are parallel; then the sufficient and necessary conditions for the equilibrium of the particle are expressed by the three following equations,

$$\Sigma (P\cos\alpha) = 0, \quad \Sigma (P\cos\beta) = 0, \quad \Sigma (P\cos\gamma) = 0,$$

where the Σ represents the summation of all such quantities as $P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$, for all the different forces of the system; or the algebraical sum of the resolved parts of all the forces of the system estimated parallel to each of the three straight lines must be equal to zero. If all the forces acting on the particle lie within a single plane, then two of the three straight lines being taken in this plane, the three equations of equilibrium will evidently be reduced to two.

The conditions for the equilibrium of a particle acted on by oblique forces, appear to have been first distinctly conceived by Stevin of Bruges[1]. He establishes by reasoning, which although indirect is satisfactory and ingenious, the ratio which the weight of a particle supported on an inclined plane bears to the force by which it is sustained, the force being supposed to act along the plane. He then announces generally, without however supplying a corresponding extension of demonstration, that the condition of equilibrium of any three forces acting on a particle, consists in the proportionality of the forces to the sides of a triangle to which they are parallel. The first rigorous

[1] **Beghinselen der Waaghconst, 1586. I. *Livre de la Statique*, prop. 19.**

demonstration of Stevin's theorem in its general form, was obtained by Roberval[1] from the nature of the lever. The idea of a triangle of equilibrium had occurred indeed somewhat earlier to Michael Varro[2], of Geneva, in application to the equilibrium of forces acting on the sides of a right-angled-triangular wedge: it does not appear, however, that Varro's notion was based upon any very distinct conception of the nature of statical pressure. The Principle of the Parallelogram of Forces, which is in fact a mere modification of Stevin's theorem, was announced almost simultaneously by Newton[3] and Varignon[4]; by whom it was inferred from the consideration of the composition of motions. In the same year was published by Lami, in a little treatise entitled *Nouvelle manière de démontrer les principaux Théorèmes des élémens des Mécaniques*, a theorem in which it is asserted, that if three forces P, Q, R, keep a particle at rest, then

$$P : Q : R :: \sin(Q, R) : \sin(R, P) : \sin(P, Q),$$

where (Q, R), (R, P), (P, Q) denote the angles between the directions of Q and R, R and P, P and Q, respectively. The virtual coincidence of this theorem with the Principle of the Parallelogram of Forces, subjected Lami to the imputation of plagiarism, an aspersion cast upon him by the author of the *Histoire des Ouvrages des Savans*, (April 1688). Lami combated this insinuation in a letter published in the *Journal des Savans*, (Sept. 13, 1688), to which the Journalist replied in the following December, when the controversy appears to have terminated. The first unexceptionable demonstration of the Parallelogram of Forces on pure statical principles, without the introduction of the idea of motion, was given by Daniel Bernoulli[5]. Many other proofs of the proposition have been since given. Eighteen demonstrations have been collected and

[1] *Traité de Mécanique*, printed in 1636, in the *Harmonie Universelle de Mersenne*, and in a work also by Mersenne, entitled *Cogitata Physico-Mathematica*, published in 1644.

[2] *Tractatus de Motu*, 1584.

[3] *Principia*, lex iii. cor. 2, 1687.

[4] *Projet de la Nouvelle Mécanique*, 1687.

[5] *Comment. Petrop.*, Tom. I. p. 126, 1726.

examined by Jacobi[1], by the following authors: D. Bernoulli, 1726; Lambert, 1771; Scarella, 1756; Venini, 1764; Araldi, 1806; Wachter, 1815; Kœstner, Marini, Eytelwein, Salimbeni, Duchayla[2]; two different proofs by Foncenex, 1760; three by D'Alembert; and those of Laplace and Poisson.

SECT. 1. *No Friction.*

(1) P and W (fig. 17) are two heavy particles; W is attached to the end of a fine thread, and P is suspended from a fixed point C of the thread; the thread has one extremity attached to a fixed point A, and passes through a smooth small ring at B in the same horizontal line with A; to find the ratio between P and W, that the vertical line through C may bisect AB in D.

From the supposition it is evident that $\angle ACD = \angle BCD$; let each of these angles be denoted by θ: let $T=$ the tension of the string CA; $CA=b$, $AB=a$; the ring B being perfectly smooth, W will be the tension of the string BC.

Hence for the equilibrium of the point C we have, resolving vertically the forces which act on it,

$$(T+W)\cos\theta = P,$$

and, resolving horizontally,

$$T\sin\theta = W\sin\theta, \quad \text{or} \quad T=W;$$

hence $$2W\cos\theta = P, \quad \cos\theta = \frac{P}{2W} \,\ldots\ldots\ldots\ldots\ldots (1);$$

but from the geometry we see that

$$b\sin\theta = \tfrac{1}{2}a, \quad \sin\theta = \frac{a}{2b} \,\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Squaring the equations (1) and (2), and adding the resulting equations, we have

$$1 = \frac{P^2}{4W^2} + \frac{a^2}{4b^2}, \quad \frac{P^2}{4W^2} = \frac{4b^2-a^2}{4b^2},$$

[1] Whewell's *Philosophy of the Inductive Sciences*, Vol. I. p. 197.

[2] *Correspondance sur l'Ecole Polytechnique*, Tom. I. p. 83, anno 1805.

and therefore $$\frac{P}{W}=\frac{(4b^2-a^2)^{\frac{1}{2}}}{b},$$

which determines the required ratio.

(2) A particle P (fig. 18) is placed on the surface of a smooth prolate spheroid, and is attracted towards the foci S and H with forces varying as SP^m and $HP^{m'}$; to find the position of equilibrium.

Draw a tangent KPL at the point P in the plane passing through the three points S, H, P; let μ, μ', be the absolute forces towards S, H; let $SP=r$, $HP=r'$. Then for the equilibrium of the particle we have, resolving forces parallel to the line KPL,

$$\mu . r^m \cos \angle SPK=\mu' r'^{m'} \cos \angle HPL;$$

but $\angle SPK=\angle HPL$, by the nature of ellipses; hence

$$\mu r^m=\mu' r'^{m'};$$

also, $2a$ denoting the axis of the spheroid, $2a=r+r'$; hence, for the determination of r and r' we have the two equations

$$\mu r^m=\mu'(2a-r)^{m'}, \qquad \mu(2a-r')^m=\mu' r'^{m'}.$$

(3) Two weights m, m', are attached to the points O, O', (fig. 19) of a string $AOO'A'$, suspended from two tacks at A and A' in the same horizontal line; to find the positions of the points that their vertical distances from the horizontal line through A and A' may have given equal values.

Draw OE, $O'E'$, vertical; let $OE=a=O'E'$, $AA'=b$, $c=$ the length of the string; $\angle AOE=\theta$, $\angle A'O'E=\theta'$; $T=$ the tension of the string OO'.

Then for the equilibrium of O we have, by Lami's Principle,

$$\frac{T}{m}=\frac{\sin(\pi-\theta)}{\sin(\frac{1}{2}\pi+\theta)}=\tan\theta;$$

and, for the equilibrium of O',

$$\frac{m'}{T}=\frac{\sin(\frac{1}{2}\pi+\theta')}{\sin(\pi-\theta')}=\cot\theta'.$$

From these two equations we get

$$m\tan\theta=m'\tan\theta' \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, from the geometry,

$$EE' = AA' - AE - A'E'$$
$$= b - a(\tan\theta + \tan\theta');$$

but we have also, from the geometry,

$$EE' = OO' = c - AO - A'O'$$
$$= c - a(\sec\theta + \sec\theta');$$

hence $a(\sec\theta - \tan\theta + \sec\theta' - \tan\theta') = c - b$...............(2).

From the equations (1) and (2) the values of θ, θ', are to be determined, and then, EO and $E'O'$ being given, AO, $A'O'$ will be known.

Diarian Repository, p. 627.

(4) A fine string is fixed at two points A and B (fig. 20) in the same horizontal line, and passes over a given set of pegs in the line AB, equal given weights being hung on between each two adjacent pegs, and also between A and B and the pegs adjacent to them: to find the position of equilibrium, and the tension of the string.

Let p, r, be any two successive pegs, pqr the corresponding portion of the string; W the magnitude of each of the weights. Let $\angle qpr = \alpha$, $T =$ the tension of the string, $c =$ the length of the piece pqr of the string, $l =$ the length of the whole string, $AB = a$.

Then, for equilibrium,

$$2T \sin\alpha = W \text{..........................(1)},$$
$$pr = c\cos\alpha \text{.....................(2)}.$$

From (1), since T is the same throughout the string, we see that α is the same for every triangle such as pqr: hence

$$\Sigma(pr) = \cos\alpha\,\Sigma(c), \quad \text{or} \quad a = l\cos\alpha \text{...........(3)}.$$

From (1), (2), (3), we have

$$T = \frac{Wl}{2(l^2 - a^2)^{\frac{1}{2}}},$$

and
$$c = \frac{l}{a} \cdot pr.$$

(5) A weight W is sustained upon a smooth inclined plane AB (fig. 21), by three forces, each equal to $\frac{1}{3}W$, one acting vertically upwards, another along AB, and the third parallel to the horizontal line AC; to find the inclination of AB to the horizon.

The required angle of inclination $= 2\tan^{-1}\frac{1}{2}$.

(6) Two forces P, Q, of known magnitudes, acting respectively parallel to the base and length of an inclined plane, will each of them singly sustain upon it a particle of weight W; to determine the magnitude of W.

$$W = \frac{PQ}{(P^2 - Q^2)^{\frac{1}{2}}}.$$

(7) Two heavy particles, P and Q, (fig. 22), are connected together by a fine thread passing over a smooth pulley at C; P rests on a smooth inclined plane AB, and Q hangs freely; to determine the position of equilibrium and the pressure on the inclined plane.

Let $\alpha =$ the inclination of the plane to the horizon, $R =$ the pressure, and $\theta =$ the angle CPB; then

$$\cos\theta = \frac{P\sin\alpha}{Q}, \qquad R = P\cos\alpha - (Q^2 - P^2\sin^2\alpha)^{\frac{1}{2}}.$$

(8) A weight W is supported on an inclined plane AB, (fig. 21), by three forces, each equal to P, one acting vertically upwards, another parallel to the horizontal line AC, and the third along AB: to find the inclination of AB.

$$\text{The required inclination} = 2\tan^{-1}\left(\frac{P}{W-P}\right).$$

(9) A particle P is placed within a thin parabolic tube AP, (fig. 23), the axis Ax of the parabola being vertical; the particle is acted on by gravity and by a force $\mu . PM$ tending from Ax, to which PM is perpendicular; to find the conditions of equilibrium.

There will be no position of equilibrium unless the latus rectum of the parabola be equal to $\frac{2g}{\mu}$; and under this condition every point of the tube will be a position of rest.

SECT. 2. *Friction.*

(1) Two heavy particles P and P' (fig. 24) rest on two inclined planes CA, $C'A$, and are connected together by a fine string passing over a smooth pulley at O in the vertical line through A; to determine the positions of P and P' when P is only just sustained.

Let a be the length of the string POP', T its tension, which will be the same throughout; W, W', the weights of the particles P, P'; μ, μ', the coefficients of friction on the planes CA, $C'A$, and R, R', their reactions; α, α', the inclinations of the two planes, and θ, θ', of the two portions of the string, to the vertical.

Then, since by hypothesis the friction on P will be exerted up CA, and that on P' down AC', we have for the equilibrium of P, resolving forces parallel and perpendicular to CA,

$$\mu R + T\cos(\alpha - \theta) = W\cos\alpha \quad\ldots\ldots\ldots\ldots (1),$$

$$R + T\sin(\alpha - \theta) = W\sin\alpha \quad\ldots\ldots\ldots\ldots (2);$$

and in the same way for the equilibrium of P',

$$\mu' R' + W'\cos\alpha' = T\cos(\alpha' - \theta') \quad\ldots\ldots\ldots\ldots (3),$$

$$R' + T\sin(\alpha' - \theta') = W'\sin\alpha' \quad\ldots\ldots\ldots\ldots (4).$$

From (1) and (2),

$$T\{\cos(\alpha - \theta) - \mu\sin(\alpha - \theta)\} = W(\cos\alpha - \mu\sin\alpha);$$

and, from (3) and (4),

$$T\{\cos(\alpha' - \theta') + \mu'\sin(\alpha' - \theta')\} = W'(\cos\alpha' + \mu'\sin\alpha').$$

Eliminating T between these two last equations, we obtain

$$W'(\cos\alpha' + \mu'\sin\alpha')\{\cos(\alpha - \theta) - \mu\sin(\alpha - \theta)\}$$

$$= W(\cos\alpha - \mu\sin\alpha)\{\cos(\alpha' - \theta') + \mu'\sin(\alpha' - \theta')\}.$$

Assume $\mu = \tan\epsilon$, $\mu' = \tan\epsilon'$; then this equation becomes

$$W'\cos(\alpha' - \epsilon')\cos(\alpha - \theta + \epsilon) = W\cos(\alpha + \epsilon)\cos(\alpha' - \theta' - \epsilon') \ldots (5).$$

Again, from the geometry, if $OA = k$,

$$OP = \frac{k\sin\alpha}{\sin(\alpha - \theta)}, \qquad OP' = \frac{k\sin\alpha'}{\sin(\alpha' - \theta')},$$

and therefore, since $OP + OP' = a$,

$$a = \frac{k \sin \alpha}{\sin (\alpha - \theta)} + \frac{k \sin \alpha'}{\sin (\alpha' - \theta')} \dots\dots\dots\dots\dots (6).$$

The angles θ, θ', are to be determined from the equations (5) and (6).

(2) Given the semi-sum and semi-difference of the greatest and least angles which the direction of a force, supporting a heavy particle on a rough inclined plane, may make with the plane, and the least elevation of the plane when the particle would, without being supported, slide down it; to determine the angle at which the same force, when inclined to a smooth plane of the same elevation, would support the same particle.

Let ϵ denote the least angle which the force may make with the rough plane to support the particle, P the magnitude of the force, R the reaction of the plane at right angles to itself, μ the coefficient of friction, α the inclination of the plane to the horizon, W the weight of the particle. Then, since the friction must in this case act down the plane, we have for the equilibrium of the particle, resolving forces parallel to the inclined plane,

$$P \cos \epsilon = \mu R + W \sin \alpha;$$

and, resolving forces at right angles to the plane,

$$P \sin \epsilon + R = W \cos \alpha.$$

Eliminating R between these two equations, we get

$$P (\cos \epsilon + \mu \sin \epsilon) = W (\mu \cos \alpha + \sin \alpha) \dots\dots\dots\dots (1).$$

Let ϕ be the least elevation of the plane for the particle without support to slide down it; then $\tan \phi$ will be equal to μ; hence from (1),

$$P = W \frac{\sin (\alpha + \phi)}{\cos (\epsilon - \phi)} \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

If ϵ' denote the greatest angle which P may make with the inclined plane consistently with the equilibrium of the particle, then the friction will act with the greatest force it can exert up

the plane; hence, making μ negative, or putting $-\phi$ for ϕ, we shall have from (2), ϵ' replacing ϵ,

$$P = W\frac{\sin(\alpha-\phi)}{\cos(\epsilon'+\phi)} \dots\dots\dots\dots\dots (3).$$

Also, if ϵ'' denote the angle of P's inclination in the case of a smooth plane of the same elevation, we have, putting $\phi = 0$ in (2), and replacing ϵ by ϵ'',

$$P = W\frac{\sin\alpha}{\cos\epsilon''} \dots\dots\dots\dots\dots (4).$$

From (2) and (3)

$$\cos(\epsilon-\phi) + \cos(\epsilon'+\phi) = \frac{W}{P}\{\sin(\alpha+\phi) + \sin(\alpha-\phi)\},$$

and therefore if $S = \frac{1}{2}(\epsilon'+\epsilon)$ and $D = \frac{1}{2}(\epsilon'-\epsilon)$, we get

$$2\cos S\cos(D+\phi) = 2\frac{W}{P}\sin\alpha\cos\phi$$

$$= 2\cos\epsilon''\cos\phi, \text{ by } (4);$$

hence $$\cos\epsilon'' = \frac{\cos S}{\cos\phi}\cos(D+\phi).$$

(3) P is the lowest point on the rough circumference of a circle in a vertical plane at which a particle can rest, friction being equal to pressure; to determine the inclination of the radius through P to the horizon.

The required angle $= \frac{\pi}{4}$.

(4) A given force P, acting parallel to the horizon, just sustains a body of given weight W on a rough inclined plane, the angle of which is θ: the same body will just rest without support on a plane of the same material, the inclination of which is α: to determine θ.

The tangent of θ is equal to

$$\frac{P + W\tan\alpha}{W - P\tan\alpha}.$$

(5) A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being α: to determine

whether it will be easier to lift the body or to drag it along the plane.

It will be easier to lift or to drag it accordingly as the coefficient of friction is less or greater than

$$\frac{\sin\left(\frac{\pi - 2\alpha}{4}\right)}{\sin\left(\frac{\pi + 2\alpha}{4}\right)}.$$

(6) A weight is supported on a rough inclined plane by a force exactly equal to it: to find the direction of the force.

If θ denote the inclination of the force to the inclined plane, α the plane's inclination, and μ the coefficient of friction,

$$\theta = \alpha - 2\beta + \tfrac{1}{2}\pi,$$

where β may have any value from $-\tan^{-1}\mu$ to $+\tan^{-1}\mu$.

CHAPTER III.

EQUILIBRIUM OF A SINGLE BODY.

LET any system of forces act upon a body consisting of a system of points rigidly connected together. Take any three straight lines OA, OB, OC, (fig. 25), in space, no two of which are parallel to each other. Let A, B, C, denote the whole resolved part, parallel to each of these straight lines, of any one of the forces of the system; A, B, C, being positive or negative quantities according as they act in the directions OA, OB, OC, or the opposite ones. Then, $\Sigma(A)$, $\Sigma(B)$, $\Sigma(C)$, denoting the algebraical sums of the resolved parts of all the forces of the system parallel to these three straight lines, it is necessary for the equilibrium of the body that we have

$$\Sigma(A) = 0, \quad \Sigma(B) = 0, \quad \Sigma(C) = 0 \ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

Again, let $O'A'$, $O'B'$, $O'C'$, be any three straight lines in space, no two of which are parallel to each other. Let any force of the system be resolved into two parts, the one at right angles to $O'A'$, and the other parallel to it; let A' be the magnitude of the part which is at right angles to $O'A'$, and a' the perpendicular distance between $O'A'$ and the direction of A'; then $A'a'$ is called the moment of A' about $O'A'$, and the sum of all such moments for all the forces of the system will be denoted by $\Sigma(A'a')$, those moments being considered positive which tend to twist the body about $O'A'$ in one direction, and those which tend to twist it in the opposite direction being considered negative. Similarly the algebraical sums of the moments about $O'B'$, $O'C'$, respectively, will be denoted by $\Sigma(B'b')$, $\Sigma(C'c')$. Then for the equilibrium of the body it is necessary that we have

$$\Sigma(A'a') = 0, \quad \Sigma(B'b') = 0, \quad \Sigma(C'c') = 0 \ldots\ldots\ldots \text{(II)}.$$

The three equations (I.), together with the three equations (II.), are universally sufficient and universally necessary for the equilibrium of any body. It may be proper to remark, that any of the lines $O'A'$, $O'B'$, $O'C'$, may be taken to coincide with any of the lines OA, OB, OC, according to convenience.

If all the forces of the system lie in one plane, then the lines OA, OB, $O'A'$, $O'B'$, being taken within this plane, and the line OC at right angles to it, the six equations of equilibrium will be reduced to the three following,

$$\Sigma(A)=0, \quad \Sigma(B)=0, \quad \Sigma(C'c')=0 \ldots\ldots\ldots\ldots(\text{III.});$$

for it is evident that the three other equations will be identically satisfied.

The basis of the general equations of equilibrium consists in the Theory of the Composition and Resolution of Forces, of which we have treated in the preceding chapter, and in the Theory of Moments. The latter theory, in the case of weights acting at right angles to the arms of a straight lever, was established by Archimedes[1]. In the year 1499, the condition of equilibrium of a force acting obliquely on a lever, and supporting a weight suspended from it, was correctly stated by Leonardo Da Vinci[2], the celebrated painter, to whom must therefore be ascribed the discovery of the theory of oblique action, investigated at a later date by Stevin, in application to the Equilibrium of a Particle. The following elegant geometrical proposition, the application of which to the general Theory of Moments depends upon the Principle of the Parallelogram of Forces, was given by Varignon in his *Nouvelle Mécanique*, sect. I, lem. XVI: If from any point whatever in the plane of a parallelogram we let fall perpendiculars upon the diagonal and upon the two sides which comprehend this diagonal, the product of the diagonal by its perpendicular is equal to the sum of the products of the two sides by their respective perpendiculars, if the point lie without

[1] Ἀρχιμήδους Ἐπιπέδων ἰσοῤῥοπικῶν ἢ κέντρα βαρῶν ἐπιπέδων τὸ Α. Πρότ. στ. Ζ.

[2] Venturi; *Essai sur les Ouvrages Physico-Mathématiques de Léonard da Vinci, avec des Fragmens tirés de ses Manuscrits apportés d'Italie*, Paris, 1797, quoted in Whewell's *History of the Inductive Sciences*, Vol. II. p. 122.

the parallelogram, or to their difference, if it lie within the parallelogram. The six general conditions of equilibrium of a system of rigidly connected points acted on by any forces whatever, were first laid down by D'Alembert, in the second chapter of his *Recherches sur la Précession des Equinoxes*, published in the year 1749.

Sect. 1. *No Friction.*

(1) A beam AB (fig. 26) rests with one end against a horizontal plane in a point A, and with the other against a vertical one in the point B; the vertical plane passing through the beam intersects at right angles the former plane in the line AC, and the latter in the line BC; the beam is attached to the point C by a string EC without weight: to find the tension of the string, E being any assigned point in the beam.

The actions of the horizontal and vertical planes upon the beam at A and B, will be in the directions AR' and BR, which are parallel respectively to CB and CA; let them be denoted by R' and R. Again, let T denote the tension of the string EC. Let G be the centre of gravity of the beam, and W its weight; then, instead of supposing the beam to have weight, we may suppose it to be a rigid rod without weight, provided that we apply the force W vertically downwards at G. Thus we have four forces, R, R', T, W, acting at four points B, A, E, G, rigidly connected together. We proceed to express the equations of equilibrium. Let $\angle ECA = \epsilon$, $\angle BAC = \alpha$, $AG = BG = a$. Then, resolving the forces parallel to the line CA, we have

$$R - T\cos\epsilon = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

resolving the forces parallel to CB, we have

$$R' - W - T\sin\epsilon = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, taking moments about the point C,

$$R \cdot 2a\sin\alpha + Wa\cos\alpha - R' \cdot 2a\cos\alpha = 0,$$

or

$$2R\sin\alpha + W\cos\alpha = 2R'\cos\alpha. \ldots\ldots\ldots\ldots(3).$$

From (1), (2), (3), there is

$$2T\cos\epsilon\sin\alpha + W\cos\alpha = 2W\cos\alpha + 2T\sin\epsilon\cos\alpha,$$

$$2T(\cos\epsilon\sin\alpha - \cos\alpha\sin\epsilon) = W\cos\alpha,$$

$$2T\sin(\alpha-\epsilon) = W\cos\alpha,$$

and therefore $$T = \frac{W\cos\alpha}{2\sin(\alpha-\epsilon)}.$$

If ϵ be equal to α, we have $T=\infty$, which shews that no tension, however great, can sustain the beam in a position of equilibrium. It is easily seen that in this case E coincides with G; and that the length of CE is sufficient to allow the beam to descend continually.

If ϵ be greater than α, T will clearly be negative; and, since the string can pull but not push, the equilibrium is impossible. Thus for the possibility of the equilibrium we must have α greater than ϵ.

(2) A smooth beam AB, (fig. 27), rests against two horizontal bars which pierce the vertical plane through the beam at right angles in the points A', A''; the beam passes under the lower and over the higher bar, its lower extremity A being sustained upon a smooth horizontal plane: to determine the pressures upon the two bars and upon the horizontal plane.

The pressures upon the bars and upon the horizontal plane will be equal to their reactions upon the beam; the reactions of the bars upon the beam will be two forces R', R'', at right angles to the beam; and the reaction of the horizontal plane will be a vertical force R. Let G be the centre of gravity of the beam; then, if we suspend its weight W from G, we may, without affecting the circumstances of the equilibrium, conceive the beam to be a rigid rod without weight. Thus we have four forces R, R', R'', W, acting respectively at four points A, A', A'', G, rigidly connected together, so as to produce equilibrium. Let $AG=a$, $A'A''=b$, and $\alpha=$ the inclination of the beam to the horizon.

Then, resolving forces parallel to the beam, we have

$$W\sin\alpha - R\sin\alpha = 0, \text{ and therefore } R = W \text{.........} (1).$$

Resolving forces at right angles to the beam,

$$R' + W\cos\alpha - R'' - R\cos\alpha = 0,$$

and therefore, by (1),

$$R' = R'' \qquad (2).$$

Again, taking moments about A,

$$R''.AA'' - R'.AA' - Wa\cos\alpha = 0,$$

and therefore, by (2),

$$R'b = Wa\cos\alpha;$$

whence $$R' = R'' = \frac{Wa\cos\alpha}{b}.$$

(3) A rigid rod AB, (fig. 28), rests upon a fixed point E, while its lower extremity A presses against a vertical line FF'; to find its position of equilibrium and also the pressures at A and E.

Let G be the centre of gravity of the rod, and W its weight; we suppose the whole weight of the rod to be collected at its centre of gravity. Let R be the reaction of the vertical line FF' upon the rod, which will be at right angles to FF'; also let R' be the reaction of the fixed point E, which will be at right angles to the rod. Let EF be at right angles to FF'; and let $EF = c$, $AG = a$, $\angle AEF = \theta$.

Then, resolving forces parallel to the rod,

$$R\cos\theta = W\sin\theta \qquad (1);$$

resolving forces at right angles to the rod,

$$R' = W\cos\theta + R\sin\theta \qquad (2);$$

and, taking moments about E,

$$\begin{aligned} R.AE\sin\theta &= W.EG\cos\theta \\ &= W(AG - AE)\cos\theta \\ &= W(a\cos\theta - c), \end{aligned}$$

and therefore $$Rc\sin\theta = W(a\cos^2\theta - c\cos\theta) \qquad (3);$$

hence, from (1) and (3),

$$Wc\frac{\sin^2\theta}{\cos\theta} = W(a\cos^2\theta - c\cos\theta)$$

$$\frac{c}{\cos\theta} = a\cos^2\theta,\ \cos\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}} \dots\dots\dots\dots\dots (4),$$

which gives the value of θ, and defines the position of the beam.

From (1) and (4) we have

$$R = W\tan\theta = W\frac{\left\{1-\left(\frac{c}{a}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}}{\left(\frac{c}{a}\right)^{\frac{1}{3}}} = W\frac{(a^{\frac{2}{3}}-c^{\frac{2}{3}})^{\frac{1}{2}}}{c^{\frac{1}{3}}},$$

which determines the pressure on the vertical line.

Also, from (1) and (2),

$$R' = W\cos\theta + W\frac{\sin^2\theta}{\cos\theta} = \frac{W}{\cos\theta},$$

and therefore, by (4), $\quad R' = W\left(\frac{a}{c}\right)^{\frac{1}{3}},$

which determines the pressure on the fixed point.

If c be greater than a, then we see by (4) that $\cos\theta$ would be greater than unity, which is impossible; thus equilibrium is impossible unless a be at least equal to c.

Fontana; *Memorie della Societa Italiana*, 1802, p. 626, &c.

(4) One end A of a beam AB, (fig. 29,) is connected to a fixed point by a hinge, about which the beam is capable of revolving in a vertical plane; the other end B is attached to a weight P by means of a string passing over a pulley C in the same vertical plane; to find the position of equilibrium.

Let a horizontal line AD through A meet a vertical line through C in the point D. Let G be the centre of gravity of the beam, at which we shall suppose its whole weight W to be collected. Produce CB to meet AE at right angles to it.

$AG=a$, $BG=b$, $AD=k$, $CD=l$, $\angle BAD=\theta$, $\phi=$ the inclination of CE to the horizon.

Then, taking moments about A,

$$P.AE = W.AF$$

or $$P(a+b)\sin(\phi-\theta) = Wa\cos\theta \dots\dots\dots\dots (1);$$

again, from the geometry,

$$(a+b)\sin\theta + BC\sin\phi = l,$$
$$(a+b)\cos\theta + BC\cos\phi = k,$$

and therefore, eliminating BC,

$$(a+b)\sin(\theta-\phi)=l\cos\phi-k\sin\phi \text{................} (2).$$

The equations (1) and (2) are sufficient for the determination of θ and ϕ, or of the position of equilibrium.

(5) A weight W (fig. 30) hangs from the end E of a rigid rod BE without weight, which rests on a smooth hinge at B, and is supported by a string CAD, passing through a fixed ring at A in the vertical line through B: the angles ACD, ADC, are equal, $\angle ABC=60°$, and $DE=BC$: to find the direction and magnitude of the pressure on the hinge.

Let X, Y, be respectively the horizontal and vertical components of the pressure exerted by the hinge on the rod, which will be equal and opposite to the components of the pressure exerted by the rod on the hinge. Let T denote the tension of the string.

The resultant of the action of the two portions of the string on the rod will evidently pass through H, the middle point of the rod, at right angles to the rod.

Hence, resolving forces parallel to the rod,

$$X\cos 30° + Y\cos 60° = W\cos 60°,$$

or

$$W-Y=X\sqrt{3} \text{......................} (1):$$

and, taking moments about H,

$$(Y+W)\,.\,\tfrac{1}{2}BE\,.\cos 30° = X\,.\,\tfrac{1}{2}BE\,.\sin 30°,$$

or

$$W+Y=\frac{X}{\sqrt{3}} \text{..........................} (2).$$

From (1) and (2),

$$X=\frac{\sqrt{3}}{2}W, \quad Y=-\tfrac{1}{2}W:$$

whence, if R denote the resultant action of the hinge on the rod,

$$R=(X^2+Y^2)^{\frac{1}{2}}=W;$$

and, if ϕ denote the inclination of R's direction to BA,

$$\tan\phi=\frac{X}{Y}=-\sqrt{3}=\tan\frac{2\pi}{3},$$

$$\phi=\tfrac{2}{3}\pi.$$

(6) A cylinder rests with its base on a smooth inclined plane; a string, attached to its highest point, and passing over a pulley at the top of the inclined plane, hangs vertically and supports a weight; the portion of the string between the cylinder and the pulley is horizontal; to determine the conditions of equilibrium.

Let P (fig. 31) be the suspended weight, W the weight of the cylinder, R the resultant action of the inclined plane on the base of the cylinder, M the point of the base through which R passes; C the centre of the base, G the centre of gravity of the cylinder. Draw GK at right angles to BB', KH horizontally to meet the vertical through G.

Let $a =$ the radius of the cylinder, $2b =$ its length, $CM = x$.

The three forces P, W, R, which act on the cylinder must pass through a single point O.

Resolving forces parallel to the inclined plane,

$$P \cos \alpha = W \sin \alpha \quad\quad (1),$$

perpendicularly to it,

$$R = P \sin \alpha + W \cos \alpha \quad\quad (2).$$

Again, from the geometry,

$$GO = GH + OH = a \sin \alpha + b \cos \alpha,$$

$$x = GO \, . \sin \alpha = \sin \alpha \, (a \sin \alpha + b \cos \alpha) \quad\quad (3).$$

Now, since x cannot be greater than a, we see by (3) that

$$a \text{ is not less than } \sin \alpha \, (a \sin \alpha + b \cos \alpha),$$

$$a \cos^2 \alpha \quad \ldots\ldots\ldots\ldots \quad b \sin \alpha \cos \alpha,$$

$$a \quad \ldots\ldots\ldots\ldots \quad b \tan \alpha \quad\quad (4).$$

The conditions (1) and (4) are sufficient and necessary for equilibrium. By (2) and (3) we know R and x.

(7) To find the force requisite to keep a door in a given position, the post being inclined at a given angle to the vertical; neglecting friction.

Let AB (fig. 32) be the door-post, $ABCD$ the door; Az a vertical line through A; Ax at right angles to Az and in the plane of BAz; E the intersection of the line CD produced with the

horizontal plane through A; join AE. With A as a centre describe a sphere cutting AB, Ax, AE, in the points p, q, r, and join these points by great circles of the sphere.

Let $\angle BAz = \beta$, and $\alpha =$ the inclination of the plane of the door to the plane xAz; $W =$ the weight of the door.

Then, since the angle pqr of the spherical triangle pqr is a right angle, we have, by Napier's rules,

$$\cos prq = \sin qpr \cos pq = \sin \alpha \sin \beta;$$

but, if ϕ denote the angle which W's direction makes with the plane of the door, it is clear that

$$\sin \phi = \cos prq;$$

hence, a denoting the distance of the centre of gravity of the door from the post, the moment of W about AB will be equal to

$$Wa \sin \alpha \sin \beta;$$

let P be the force applied at right angles to the door, at a point distant from the door-post by a space b, sufficient to keep it in its present position; then, by the equation of moments, we have

$$Pb = Wa \sin \alpha \sin \beta$$

Another solution:

The component of W in the plane zAx at right angles to AB is equal to $W \sin \beta$, and the component of $W \sin \beta$ at right angles to the door is $W \sin \beta \sin \alpha$: hence the moment of W about AB is equal to

$$Wa \sin \alpha \sin \beta,$$

and therefore

$$Pb = Wa \sin \alpha \sin \beta.$$

(8) A uniform bar AB (fig. 33) is placed in the straight line joining two centres of force K, L, which attract with forces varying directly as the distance; to find the position in which the bar will rest.

Let μ, μ', be the absolute forces of the centres K, L; let P be any point in the bar AB; $KA = x$, $LB = y$, $KP = s$, $BP = s'$, $AB = 2a$, $KL = l$; $\rho =$ the density of the bar, $\kappa =$ the

LP

area of a transverse section. Then for the equilibrium of the bar we must have

$$\int_x^{x+2a} \kappa\rho\mu s\, ds = \int_y^{y+2a} \kappa\rho\mu' s'\, ds';$$

or, since κ and ρ are supposed to be the same for all points of the bar,

$$\mu \int_x^{x+2a} s\, ds = \mu' \int_y^{y+2a} s'\, ds',$$

$$\mu \{(x+2a)^2 - x^2\} = \mu' \{(y+2a)^2 - y^2\},$$

$$\mu (x+a) = \mu' (y+a) = \mu' (l-a-x),$$

$$(\mu + \mu')\, x = \mu' l - (\mu + \mu')\, a,$$

$$x = \frac{\mu' l}{\mu + \mu'} - a;$$

similarly $$y = \frac{\mu l}{\mu + \mu'} - a.$$

The value of x, or of y, determines the position of equilibrium.

(9) A uniform beam AB, (fig. 34), of which one end A is placed upon a smooth horizontal plane OA, and of which the other end B presses against a vertical plane OB, is attracted by a centre of force situated in the point O, the intensity of the force varying directly as the distance; to determine the position of equilibrium.

Conceive the beam to be inclined at an angle ω to the horizon. Take P any point in the beam and join OP.

Let $OP = r$, $AP = s$, $AG = BG = a$, $\angle POA = \theta$, $\mu =$ the absolute force of attraction, R, R', the reactions of the planes at A, B. Then, resolving forces horizontally, we have

$$R' = \int \mu r ds \cos\theta = \mu \int_0^{2a} ds\, (2a - s) \cos\omega = 2\mu a^2 \cos\omega \ldots\ldots(1).$$

Resolving forces vertically,

$$R - W = \int \mu r ds \sin\theta = \mu \int_0^{2a} \sin\omega\, s ds = 2\mu a^2 \sin\omega \ldots\ldots\ldots(2).$$

Taking moments about O,

$$R \, . \, 2a \cos \omega = Wa \cos \omega + R' \, . \, 2a \sin \omega,$$

$$2R \cos \omega = W \cos \omega + 2R' \sin \omega \text{............} \quad (3),$$

and therefore, substituting in this equation the values of R' and R from (1) and (2), we have

$$2 \cos \omega \, (W + 2\mu a^2 \sin \omega) = W \cos \omega + 2 \sin \omega \, . \, 2\mu a^2 \cos \omega,$$

and therefore $\qquad W \cos \omega = 0, \quad \omega = \frac{1}{2}\pi,$

or the beam lies in contact with the vertical plane OB.

It is evident, however, that this is not the only position of equilibrium; the beam will plainly remain at rest if it be placed in contact with the horizontal plane OA with one extremity at O. In writing down the equations (1), (2), (3), it is tacitly assumed that the beam receives no pressure from the planes excepting at its extremities, an hypothesis which holds good in the former position of equilibrium while it evidently does not in the latter: it is for this reason that, in our analytical investigation, out of the two admissible values 0 and $\frac{1}{2}\pi$ for ω we obtained only the latter.

(10) A rigid rod AB (fig. 35) rests against a smooth vertical wall CD, and has its lower extremity A attached to a hinge about which it can revolve freely; to find the pressure on the wall and upon the hinge.

Let G be the centre of gravity of the rod, at which we may suppose its whole weight W to be collected; let $AG = b$, $AB = a$, $\angle BAC = \alpha$. Also let R denote the reaction of the wall against the rod, which will take place at right angles to CD; and let R', S, be the vertical and horizontal parts of the reaction of the hinge upon the rod. Then

$$R = \frac{Wb}{a \tan \alpha} = S, \qquad R' = W.$$

This problem was first proposed under a vicious form in a work by Stone; where the author proposes to determine the position of AB corresponding to a maximum value of R. In a

review of Stone's work by John Bernoulli[1], the solution given by Stone was declared to be erroneous, and a different one was offered by the reviewer. Bernoulli's solution is, however, essentially vicious. The problem was correctly solved for the first time by Couplet[2]. The opinions however, both of mathematicians and of architects, were for many years divided as to the respective merits of the solutions given by Bernoulli and by Couplet, and even down to very late years numerous memoirs have appeared on the subject by different mathematicians, with various conclusions; several of whom have arrived at results at variance with the solutions both of Bernoulli and of Couplet. The reader who may be curious to examine the various solutions of this problem, which by the aberrations of the learned rather than by any intrinsic difficulty has obtained considerable celebrity, is referred to a memoir by Franchini in the *Memorie della Società Italiana*, Tom. XVI. parte 1, p. 228; 1813.

(11) A ladder of uniform thickness rests with its lower end upon a smooth horizontal plane, and its upper end on a slope inclined at an angle of 60° to the horizon; the ladder makes an angle of 30° with the horizon: to find the force which must act horizontally at the foot of the ladder to prevent sliding.

If W denote the weight of the ladder,

$$\text{the required force} = \frac{3^{\frac{1}{2}}}{4} W.$$

(12) A sphere rests upon two inclined planes; to find the pressure experienced by each.

Let W be the weight of the sphere; α, α', the inclinations of the inclined planes to the horizon; and R, R', their respective pressures. Then

$$R = \frac{W \sin \alpha'}{\sin (\alpha + \alpha')}, \quad R' = \frac{W \sin \alpha}{\sin (\alpha + \alpha')}.$$

Leibnitz; *Opera*, Tom. III. p. 176.

(13) A sphere, of which C is the centre, is supported on an inclined plane AB by a string CB which is horizontal; to find the tension of CB.

[1] *Opera*, Tom. IV. p. 189. [2] *Mémoires de l'Académie de Paris*, 1731, p. 69.

If W be the weight of the sphere, and α the inclination of the plane to the horizon,

$$\text{the tension of the string} = W \tan \alpha.$$

(14) A given weight P is suspended from the rim of a uniform hemispherical bowl placed on a horizontal plane; to find the position in which the bowl will rest.

If W denote the weight of the bowl, c the distance between its centre and its centre of gravity, and θ the inclination of the axis of the bowl to the vertical,

$$\tan \theta = \frac{Pr}{Wc}.$$

(15) A rigid rod without weight passes through two fixed rings, and is urged by a force P in the direction of its length against a plane to which it is inclined at an angle α: to find the pressure on the plane.

The required pressure is equal to $P \operatorname{cosec} \alpha$.

(16) One end of a beam, the weight of which is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane, inclined at an angle α to the horizon; the string, passing over a pulley at the top of the inclined plane, hangs vertically, supporting a weight P: to find the condition of equilibrium.

If $a =$ the length of the beam, and $b =$ the distance of its centre of gravity from its lower end, the condition of equilibrium is expressed by the equation

$$Pa = Wb \sin \alpha,$$

which shews that, if the beam can rest in any one position, it will rest in all positions.

(17) A uniform beam rests upon two perfectly smooth inclined planes; to find its position and its pressure upon the two planes.

Let α, α', be the inclinations of the two planes to the horizon; R, R', the pressures which they experience; then, supposing the end of the beam which rests against the former plane to be the

lower one, and θ to be the inclination of the beam to the horizon, we shall have, W being the weight of the beam,

$$\tan\theta = \frac{\sin(\alpha' - \alpha)}{2\sin\alpha'\sin\alpha}, \quad R = \frac{W\sin\alpha'}{\sin(\alpha + \alpha')}, \quad R' = \frac{W\sin\alpha}{\sin(\alpha + \alpha')}.$$

(18) A uniform beam ABC (fig. 36) is placed with one end A in a fixed hemispherical bowl, and, being of greater length than the diameter of the bowl, rests upon the rim of the bowl at the point B; to find the position in which the beam will rest, the radius OB of the bowl being horizontal.

If r be the radius of the bowl, $2a$ the length of the beam, and θ its angle of inclination to the horizon; then

$$4r\cos^2\theta - a\cos\theta - 2r = 0.$$

(19) To find the position of equilibrium of a uniform beam, one end of which rests against a vertical plane, and the other on the interior surface of a given hemisphere.

Let r be the radius of the hemisphere, c the distance of its centre from the vertical plane, $2a$ the length of the beam; θ the inclination of the beam to the horizon, and ϕ of the radius at the point where the beam presses against the hemisphere. Then the position of equilibrium will depend upon the equations

$$\tan\phi = 2\tan\theta, \quad 2a\cos\theta = r\cos\phi + c.$$

(20) A beam AB (fig. 37) leans against a smooth vertical prop CD, the end A being prevented from sliding along the horizontal plane AD by a string AD fastened at D; to find the tension of the string.

Let G be the centre of gravity of the beam; $AG = a$, $CD = b$, $AD = c$, $W =$ the weight of the beam, $T =$ the tension; then

$$T = \frac{abc}{(b^2 + c^2)^{\frac{3}{2}}} W.$$

(21) A uniform rigid rod AB (fig. 38) rests upon a fixed point E, while its lower end A presses against a vertical line FF'; a weight P is suspended from the extremity B; to find its position of equilibrium.

Let W = the weight of the rod, b = the perpendicular distance of E from the line FF', $AE = x$, a = the length of the rod; then

$$x = \left(ab^2 \frac{P + \frac{1}{2}W}{P + W}\right)^{\frac{1}{3}}.$$

Fontana; *Memorie della Societa Italiana*, 1802, p. 630.

If we suppose $W = 0$, then we shall have $x = (ab^2)^{\frac{1}{3}}$, whatever be the magnitude of P. This problem is discussed by Euler, *Acad. des Sciences de Berlin*, Tom. VII. p. 196, in illustration of Maupertuis' Principle of Rest.

(22) One end of a beam is connected with a horizontal plane by a hinge about which the beam can revolve freely in a vertical plane; the other end is attached to a weight by means of a string passing over a pulley in the same vertical plane; to find the position of equilibrium.

Let a, b, be the distances of the centre of gravity of the beam from its lower and its higher extremities, W its weight, and θ its inclination to the horizon; let ϕ be the inclination of the string to the horizon, and P the weight attached to its extremity; let l be the distance of the pulley from the horizontal and k from the vertical line through the hinge. Then the position of equilibrium will depend upon the equations

$$P(a+b)\sin(\phi - \theta) = Wa\cos\theta,$$

$$(a+b)\sin(\phi - \theta) = k\sin\phi - l\cos\phi.$$

(23) A uniform beam rests with one end upon a given inclined plane, the other end being suspended by a string from a fixed point above the plane; to determine the position of equilibrium, the tension of the string, and the pressure on the plane.

Let $2a$ be the length of the beam, θ its inclination to the inclined plane, W its weight, and R the pressure which it exerts on the inclined plane; let T be the tension of the string, c its length, and ϕ its inclination to the inclined plane; also let b be the distance of the fixed point from the plane; and α the inclination of the plane to the horizon.

Then the position of the beam will depend upon the two equations

$$2 \sin (\phi - \theta) \sin \alpha = \cos \phi \cos (\theta + \alpha),$$
$$c \sin \phi + 2a \sin \theta = b;$$

and then R and T will be given by the formulæ

$$R = \frac{W \cos (\alpha + \phi)}{\cos \phi}, \quad T = \frac{W \sin \alpha}{\cos \phi}.$$

(24) A uniform beam rests with one end against a smooth vertical plane, its other end being supported by a string attached to a fixed point in the plane; to determine the position of the beam, its pressure against the plane, and the tension of the string.

Let b be the length and T the tension of the string; $2a$ the length of the beam, W its weight, and R its pressure against the vertical plane; also let ϕ, θ, be the inclinations of the beam and of the string to the vertical. Then

$$\sin \theta = \left(\frac{16a^2 - b^2}{3b^2}\right)^{\frac{1}{2}}, \quad \sin \phi = \left(\frac{16a^2 - b^2}{12a^2}\right)^{\frac{1}{2}},$$
$$T = \frac{3^{\frac{1}{2}} b W}{(4b^2 - 16a^2)^{\frac{1}{2}}}, \quad R = \left(\frac{16a^2 - b^2}{4b^2 - 16a^2}\right)^{\frac{1}{2}} . W.$$

(25) A weight W hangs from a rod BC, (fig. 39), which rests on a fulcrum at B, and is supported by a string DA at right angles to the rod, D being the middle point of BC; to determine the magnitude and direction of the pressure on the fulcrum, the rod being inclined to the horizon at an angle of 30°, and being without weight.

Let $BD = CD = a$; and let X, Y, represent the vertical and horizontal components of the pressure exerted by the rod on the fulcrum; then

$$X = \tfrac{1}{2} W, \quad Y = \frac{3^{\frac{1}{2}}}{2} W;$$

and, if ϕ denote the inclination of the resultant pressure to the vertical, and R its magnitude,

$$R = W, \quad \phi = \tfrac{1}{3}\pi.$$

(26) A uniform beam AB (fig. 40) moveable in a vertical plane about a hinge at A, leans upon a prop CD situated in the same plane; to determine the strain upon the prop CD.

Let $AB = 2a$, $CD = b$, $\angle BAC = \alpha$, $\angle ACD = \beta$. Then the resolved part of the pressure of AB on CD at right angles to CD, which measures the strain on the prop, will be equal to

$$\frac{Wa \sin 2\alpha \cos(\alpha + \beta)}{2b \sin \beta}.$$

(27) A uniform beam is hung from a fixed point by two unequal strings attached to its extremities: to compare the tension of each string with the weight of the beam.

Let a, b, represent the lengths of the strings, P, Q, their respective tensions, c the length and W the weight of the beam;

then $$\frac{P}{W} = \frac{a}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}, \quad \frac{Q}{W} = \frac{b}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}.$$

(28) An isosceles right-angled triangle rests in a vertical plane with the right angle downwards, between two pegs at a distance a from each other in the same horizontal line; to determine its positions of equilibrium.

Let h = the perpendicular from the right angle on the base, and θ = the inclination of the base to the horizon; then

$$\theta = 0, \text{ or } \theta = \cos^{-1}\left(\frac{h}{3a}\right).$$

(29) A flat board DE, (fig. 41), in the form of a square, is supported upon two fixed points P, Q, with its plane vertical, the distance between P, Q, being equal to half a side of the square: to find the positions of equilibrium.

If α = the inclination of PQ, and θ of AE to the horizon, the positions of equilibrium are given by the equation

$$\sin^2(2\theta + \alpha) = \sin 2\theta.$$

(30) A uniform rod of given length rests against a peg at the focus of a parabola, the axis of which is vertical and of which the vertex is the lowest point, the lower extremity of the rod being supported on the curve; to determine the angle which the rod makes with the axis of the parabola.

If a be the length of the rod, and $4m$ the latus rectum of the parabola; then

$$\text{the required angle} = 2 \cos^{-1}\left(\frac{m}{a}\right)^{\frac{1}{3}}.$$

(31) A uniform isosceles triangle is placed within a smooth hemispherical bowl, its three angles touching the bowl; to find the position in which it will rest.

Let a = the length of each of the equal sides, h = the altitude of the triangle, r = the radius of the hemisphere, θ = the inclination of the triangle to the vertical; then

$$\tan\theta = \frac{3(4r^2h^2 - a^4)^{\frac{1}{2}}}{4h^2 - 3a^2}.$$

(32) A uniform circular lamina is placed with its centre upon a prop; to find at what points on its circumference three weights w_1, w_2, w_3, must be attached that it may remain at rest in a horizontal position.

Let (w_2, w_3), (w_3, w_1), (w_1, w_2), denote the angles at the centre of the lamina between the distances of w_2, w_3; w_3, w_1; w_1, w_2; respectively. Then

$$\cos(w_2, w_3) = \frac{w_1^2 - w_2^2 - w_3^2}{2w_2w_3}, \quad \cos(w_3, w_1) = \frac{w_2^2 - w_3^2 - w_1^2}{2w_3w_1},$$

$$\cos(w_1, w_2) = \frac{w_3^2 - w_1^2 - w_2^2}{2w_1w_2}.$$

(33) A hemisphere is fixed with its base on the ground between two parallel vertical planes, both of which touch it, and of which one reaches to a height equal to the diameter: a beam of given length and weight, supported by the hemisphere, rests over the top of the finite plane, one of its ends pressing against the indefinite plane: to find the pressures of the beam on the planes and hemisphere, and to determine the greatest length of the beam for which there can exist any pressure on the hemisphere.

Let r = the radius of the hemisphere, $2a$ = the length of the beam, R = the pressure on the top of the finite plane, S = the pressure on the indefinite plane, T = the pressure on the sphere. Then, W denoting the weight of the beam,

$$R = \frac{32a - 25r}{80r} \cdot W, \qquad S = \tfrac{3}{4}W,$$

$$T = \frac{125r - 32a}{80r} \cdot W.$$

SECT. 2. *Friction.*

Statical friction consists in the resistance arising from mutual roughness, which is opposed to the production of relative motion between two substances in contact. If the substances were perfectly smooth, their mutual pressure at every point of the surfaces of contact would take place in some determinate straight line depending upon the forms of the surfaces; if the consideration of roughness be introduced, the force of friction when called into play will exert itself at each point in a direction at right angles to the mutual pressure corresponding to perfect smoothness. The estimation of the magnitude of friction for assigned substances and for given surfaces of contact, can be effected solely by experiment.

Suppose R to denote the total pressure of two substances, of which the surfaces of contact are two planes, and let F be the greatest force which friction can exert in the prevention of relative motion; then F is taken as the measure of the statical friction. After the performance of numerous experiments, Amontons[1], who was the first to discuss scientifically the subject of friction, was led to conclude that, so long as the substances remain the same, F varies directly as R, and is independent of the magnitude of the area of contact. Thus, μ denoting some constant quantity, the magnitude of which is to be obtained by experiment, we should have for any assigned substances

$$F = \mu R,$$

where μ is called the coefficient of friction. This relation, however, although generally adopted by mathematicians, is probably not quite accurate. Muschenbroek[2], and the Abbé Nolet[3], concluded from experiments that the value of μ depends in some degree upon the magnitude of the area of contact, and that for an assigned area of contact it does not remain invariable for all

[1] *Mémoires de l'Académie des Sciences de Paris*, 1699, p. 206.
[2] *Introduct. ad Phil. Nat.* Tom. I. cap. 9, 1762. *Lect. Phys. Exp.* Tom. I. p. 241.
[3] *Leçons de Physique Expérimentale*, Tom. I. p. 230; 1754.

values of R. Bossut[1] agreed with Amontons in supposing μ to be independent of the area of contact, but considered that its value decreases as R increases. Various experimenters have exerted their labours on the same subject with very different conclusions. Professor Vince[2] concluded, after the performance of very careful experiments, that the coefficient of friction does really diminish with the increase of R, and that for a given pressure it decreases when the area of contact is diminished. It would appear however, from the valuable experiments of Coulomb[3] and Ximenes[4], that the variation of μ, owing to any change in the magnitude of the area of contact, is extremely small and of an irregular character, and that it decreases very slightly as R increases. Bossut[1] has remarked, that the statical friction between two substances becomes greater by allowing them to remain in contact for some time before it is called into play, an observation which has been fully confirmed by the experiments of Coulomb.

If the surfaces of contact be not plane areas, the coefficient of friction will on this account receive a change of value; and generally it will depend upon the forms of the surfaces of contact, as well as upon the nature of the substances. The friction of a solid cylinder against a hollow one has been considered by Coulomb and Ximenes, who have found it to be much smaller than between two plane surfaces of the same substance; the coefficient of friction is, however, approximately constant, as in the case of plane surfaces of contact.

The friction of which we have been speaking, is the friction called into play by the *rubbing* of two substances against each other; the roughness of substances, however, exerts force to interrupt the production of relative motion also in the case when one body is urged to *roll* along another without *rubbing;* this may be called the friction of cohesion, depending probably upon the mutual tenacity of the particles of the two bodies. This species of friction was first noticed by Bossut, and after-

[1] *Traité de Mécanique*, Part I. chap. 4, sect. 1, p. 178.
[2] *Philosophical Transactions*, 1785, Part I. p. 165.
[3] *Mémoires présent. à l'Academie*, Tom. x. 1785.
[4] *Terria e Pentica delle Resist. de' Sol. ne' loro Attr.* Pisa, 1782.

wards carefully investigated by Ximenes and Coulomb: in the case of a cylinder rolling along a plane, the friction of cohesion is found to vary inversely as the diameter.

The friction which exists between two substances in motion, which may be called their dynamical friction, is very considerably less than their statical friction. The dynamical friction is measured by the force necessary to keep the bodies in motion; the statical friction by the force necessary to set them originally in motion. The difference of the magnitudes of statical and dynamical friction was noticed by Camus[1] and Desaguliers[2], and afterwards by various other experimenters. Professor Vince ascertained by experiments, that dynamical friction is a constant force for hard substances, whatever be the velocity of the relative motion; but that in the case of softer bodies it increases considerably with an increase of velocity. The friction of pivots has been fully considered by Coulomb in the *Mémoires de l'Acad. des Sciences de Paris*, 1790. The friction and rigidity of ropes was first investigated experimentally by Amontons in the memoir to which we have alluded above, and afterwards by Coulomb and Ximenes.

(1) A uniform beam AB, (fig. 42), resting with one end A upon a rough horizontal plane KL, has its other end B attached to a string which passes over a smooth pulley E, and supports a weight P; to determine the range of positions in which the beam may be placed consistently with equilibrium.

Let G be the centre of gravity of the beam, and W its weight; θ, ϕ, the angles of inclination of AB, BE, to the horizon for any position of equilibrium; let $AG = BG = a$; let F denote the friction, estimated along LK, which is called into play at A, and which will be at right angles to R the vertical reaction of the plane on the beam. Suppose the whole weight of the beam to be collected at its centre of gravity.

Then for the equilibrium of the beam we have, resolving the forces horizontally,

$$F = P\cos\phi \dots\dots\dots\dots\dots\dots (1);$$

[1] *Traité des Forces Mouvantes.* [2] *Cours de Physique.*

resolving vertically,

$$R + P \sin\phi = W \text{ .... } (2);$$

and, taking moments about A,

$$Wa \cos\theta = P \cdot 2a \sin(\phi - \theta),$$

or

$$W \cos\theta = 2P \sin(\phi - \theta) \text{ .... } (3).$$

Assume $F = \lambda R$, where, if μ denote the coefficient of friction between the end of the beam and the plane, λ may have any value from zero up to μ. Then by (1) we have

$$\lambda R = P \cos\phi \text{ .... } (4).$$

From (2) and (4), we obtain

$$P \cos\phi + \lambda P \sin\phi = \lambda W,$$

or, putting $\lambda = \tan\epsilon$,

$$P \cos(\phi - \epsilon) = W \sin\epsilon,$$

which determines the angle ϕ in terms of W, P, ϵ; and then θ may be determined from (3). By giving then to ϵ any values from zero up to $\tan^{-1}\mu$, we shall obtain a series of positions of equilibrium.

Suppose for instance λ to be equal to zero; then from (4)

$$P \cos\phi = 0, \text{ and therefore } \phi = \tfrac{1}{2}\pi;$$

hence, by (3), $\qquad W \cos\theta = 2P \cos\theta,$

and therefore either $W = 2P$, in which case θ remains indeterminate and may have any value whatever, or $\theta = \frac{1}{2}\pi$. Again from (2), since $\phi = \frac{1}{2}\pi$, we have

$$R = W - P \text{ .... } (5),$$

and therefore, if θ be not equal to $\frac{1}{2}\pi$, we must have

$$R = P = \tfrac{1}{2} W.$$

Thus we see that the end B of the beam must be in the vertical line through E; and that, unless AB be placed vertically, the weight P must be equal to half the weight of the beam. If the beam be placed vertically, it is clear from (5) that P may have any value from 0 up to W, but no greater value, because R cannot be negative.

If, instead of taking $\lambda = 0$, we were to give it any other value between 0 and μ, we should have to determine the values of θ and ϕ as in the present case.

(2) A beam AB (fig. 43) is supported on a prop CD by a given force P acting at a given angle of inclination to the horizon; to find the position of the beam when it is upon the point of sliding past the point C from A towards B, the prop and beam being relatively rough.

Produce BA, PA, to meet the horizontal line KL in the points F, E; let G be the centre of gravity of the beam. Let $AG = a$, $CG = x$, $\angle PEL = \alpha$, $\angle AFE = \theta$, $R =$ the reaction of the prop at right angles to AB, and μ the coefficient of friction; then μR will be the friction, of which BA is the direction.

Then for the equilibrium of the beam we have, resolving forces vertically,

$$P \sin \alpha + R \cos \theta = W + \mu R \sin \theta \quad \ldots\ldots (1);$$

resolving horizontally,

$$P \cos \alpha = R \sin \theta + \mu R \cos \theta \quad \ldots\ldots (2);$$

and, taking moments about C,

$$Wx \cos \theta = P(a + x) \sin (\alpha - \theta) \quad \ldots\ldots (3).$$

From the equations (1) and (2) there is

$$\frac{\cos \theta - \mu \sin \theta}{\sin \theta + \mu \cos \theta} = \frac{W - P \sin \alpha}{P \cos \alpha},$$

and therefore

$$P \cos \alpha (1 - \mu \tan \theta) = (W - P \sin \alpha)(\tan \theta + \mu),$$

$$P(\cos \alpha + \mu \sin \alpha) - \mu W = \{W + P(\mu \cos \alpha - \sin \alpha)\} \tan \theta;$$

assume $\mu = \tan \epsilon$; then, multiplying both sides of the equation by $\cos \epsilon$,

$$P \cos (\epsilon - \alpha) - W \sin \epsilon = \{P \sin (\epsilon - \alpha) + W \cos \epsilon\} \tan \theta,$$

$$\tan \theta = \frac{P \cos (\epsilon - \alpha) - W \sin \epsilon}{P \sin (\epsilon - \alpha) + W \cos \epsilon},$$

which determines the inclination of the beam to the horizon.

Knowing θ we may determine x from the equation (3); and thus the position of the beam will be completely ascertained.

If the beam be on the point of sliding in a direction opposite to that which we have supposed, the quantity μ must be replaced by $-\mu$, or ϵ by $-\epsilon$; and the formulæ for the former case will all become adapted to the latter.

(3) A uniform rectangular board $KLMN$, (fig. 44), is placed upon a rough inclined plane AB; supposing the inclination of the plane AB to the horizon to be gradually increased, to find whether the equilibrium of the board will be disturbed by the commencement of a rolling or of a sliding motion.

First suppose that the board begins to slide; let R be the whole of the reaction of the plane at right angles to itself on the board, μ the coefficient of friction, and ϕ the inclination of the plane at the commencement of sliding. Then, resolving forces parallel to the inclined plane,

$$\mu R = W \sin \phi;$$

and, resolving forces at right angles to it,

$$R = W \cos \phi;$$

hence, eliminating R,

$$\tan \phi = \mu.$$

Next suppose that the board tumbles over the corner K before the commencement of sliding; then the vertical through G will pass through K when ϕ has received the proper value; draw GH at right angles to the plane, let $HK = a$, $GH = b$; then

$$\tan \phi = \tan \angle KGH = \frac{a}{b}.$$

Hence, if μ be less than $\frac{a}{b}$, sliding will take place before rolling; on the contrary, if μ be greater than $\frac{a}{b}$, rolling will take place before sliding; if μ be equal to $\frac{a}{b}$, rolling and sliding will take place simultaneously.

(4) A beam PQ, (fig. 45), which is capable of free motion in every direction about a smooth hinge at P, rests with its end Q

against a rough vertical plane ABC; to determine the position of the beam when it is bordering on motion.

From P draw PO at right angles to the plane ABC; join OQ; the locus of Q will be a circle in the vertical plane having O for its centre; let G be the centre of gravity of the beam; PHV be the projection on the horizontal plane through PO of the line PGQ, H and V being the projections of G and Q; draw HK at right angles to PO; let W be the weight of the beam, μ the coefficient of friction between the beam and the vertical plane, and R their mutual pressure; μR will act in the tangent to the locus of Q at the point Q, that is, at right angles to OQ and in the plane ABC, and from A towards B;

let $\qquad PG = a,\ QG = b,\ \angle\, QPO = \alpha,\ \angle\, QOA = \theta.$

Then for the equilibrium of the beam we have, taking moments about PO,

$$W.\,HK = \mu R\,.\,OQ \qquad\qquad (1);$$

and, taking moments about the horizontal line through P, which is at right angles to PO, it being observed that the vertical resolved part of μR is $\mu R \cos \angle\, QOV$,

$$W.\,PK = R.\,QV + \mu R.\,PO \cos \angle\, QOV \qquad\qquad (2).$$

Now, from the geometry,

$$HK = GK \cos\theta = a \sin\alpha \cos\theta,$$

$$OQ = (a+b)\sin\alpha,\quad PO \cos \angle\, QOV = (a+b)\cos\alpha\cos\theta,$$

$$PK = a\cos\alpha,\quad QV = OQ\sin\theta = (a+b)\sin\alpha\sin\theta;$$

hence, from the equations (1) and (2),

$$Wa \sin\alpha\cos\theta = \mu R\,(a+b)\sin\alpha,$$

and $\quad Wa\cos\alpha = R\,(a+b)\sin\alpha\sin\theta + \mu R(a+b)\cos\alpha\cos\theta;$

dividing the latter of these equations by the former,

$$\frac{\cos\alpha}{\sin\alpha\cos\theta} = \frac{\sin\alpha\sin\theta + \mu\cos\alpha\cos\theta}{\mu\sin\alpha},$$

$$\mu\cos\alpha = \cos\theta\,(\sin\alpha\sin\theta + \mu\cos\alpha\cos\theta),$$

$$\mu\cos\alpha\sin^2\theta = \sin\alpha\sin\theta\cos\theta,$$

$$\mu\tan\theta = \tan\alpha,$$

$$\tan\theta = \frac{1}{\mu}\tan\alpha.$$

We may solve this problem also in the following manner: taking moments about the vertical line through P we have, since $\mu R\sin\theta$ is the horizontal resolved part of μR,

$$R\,.\,OV = \mu R\,.\sin\theta\,.\,PO,$$

and therefore $$OQ\cos\theta = \mu\sin\theta\,.\,PO;$$

but $$OQ = OP\tan\alpha,$$

hence $$\tan\alpha\cos\theta = \mu\sin\theta,\quad \tan\theta = \frac{1}{\mu}\tan\alpha.$$

(5) A beam AB (fig. 46) is placed with one end upon a rough horizontal plane Ox, and rests against a rough plane curve KPL at any point P; supposing that, whatever be the point P against which the beam leans, it is always in an equilibrium bordering on motion, and that the coefficient of friction is the same both for the curve and for the horizontal plane, to find the nature of the curve.

Draw PM at right angles to Ox; let G be the centre of gravity of the beam, W its weight, $AG = a$, $\angle BAx = \theta$, $OM = x$, $PM = y$, $\mu =$ the coefficient of friction; let R and R' be the normal reactions of the curve and of the plane against the beam; in consequence of friction the curve will exert on the beam a force μR along PB, and the horizontal plane a force $\mu R'$ along Ax.

Hence for the equilibrium of the beam, resolving forces parallel to Ox,

$$R\sin\theta = \mu R\cos\theta + \mu R',$$

$$R(\sin\theta - \mu\cos\theta) = \mu R' \text{........................}(1);$$

resolving forces perpendicularly to Ox,

$$R\cos\theta + \mu R\sin\theta + R' = W,$$

$$R(\cos\theta + \mu\sin\theta) + R' = W \text{..................}(2);$$

and, taking moments about A,

$$R\,.\,AP = Wa\cos\theta, \text{ or } R\,.\,AM = Wa\cos^2\theta \text{..........}(3).$$

From (1) and (2) we get

$$(1+\mu^2)R\sin\theta = \mu W,$$

and therefore, from (3),

$$(1+\mu^2)\, Wa \sin\theta \cos^2\theta = \mu W . AM,$$
$$(1+\mu^2)\, a \sin\theta \cos^2\theta = \mu . AM;$$

but $$\sin\theta = \frac{dy}{ds},\ \cos\theta = \frac{dx}{ds},\quad AM = y\frac{dx}{dy};$$

hence we have

$$(1+\mu^2)\, a \frac{dy}{ds}\frac{dx^2}{ds^2} = \mu y \frac{dx}{dy},$$

$$a(1+\mu^2)\frac{dy^2}{dx^2} = \mu y \frac{ds^3}{dx^3};$$

put $\mu = \tan\epsilon$, and this equation becomes

$$\frac{2a}{\sin 2\epsilon}\frac{dy^2}{dx^2} = y\frac{ds^3}{dx^3};$$

which is the differential equation to the curve.

If the friction of the curve and the plane be different, we may obtain the differential equation to the curve with equal ease.

(6) A uniform rod passes over the fixed point A and under the fixed point B, (fig. 47), and is kept at rest by the friction at the points A and B; to determine the circumstances of equilibrium.

Let μR, μS, be the forces of friction at A, B, respectively, R and S being the normal actions of the fixed points on the rods. Let G be centre of gravity of the rod.

Let $AB = a$, $\alpha =$ the inclination of AB to the horizon, $2b$ = the length of the rod, and $AG = x$.

Resolving forces along the rod, we have

$$\mu(R+S) = W \sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

resolving perpendicularly to the rod, we have

$$R = W\cos\alpha + S \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, taking moments about G,

$$Rx = S(x+a) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2),

$$2\mu S = W(\sin\alpha - \mu\cos\alpha) \qquad (4).$$

From (2) and (3) there is

$$aS = xW\cos\alpha \qquad (5);$$

and therefore, by (4) and (5),

$$x \,.\, 2\mu\cos\alpha = a(\sin\alpha - \mu\cos\alpha),$$

$$x = \frac{a}{2\mu}(\tan\alpha - \mu) \qquad (6).$$

Since S cannot be negative, therefore, by (5), x cannot be negative. Moreover, from the geometry, it is plain that

$$x \angle b - a \qquad (7).$$

Let λ be the coefficient of friction. Then μ may have any value between 0 and λ which gives to x, as determined by the equation (6), a positive value consistent with the inequality (7).

If $$\frac{a}{2\lambda}(\tan\alpha - \lambda) > b - a,$$

or $$\lambda < \frac{a\tan\alpha}{2b-a},$$

equilibrium is impossible.

(7) A beam rests with its lower extremity on a horizontal, and its higher against a vertical plane; having given its length, the position of its centre of gravity, and the coefficients of the friction of the horizontal and of the vertical plane, to find its position when in a state bordering on motion.

If a, b, be the distances of the centre of gravity of the beam from its lower and higher extremity; μ, μ', the coefficients of friction between the beam and the horizontal, and between the beam and the vertical plane; and θ the inclination of the beam to the horizon; then

$$\tan\theta = \frac{a - \mu\mu' b}{\mu(a+b)}.$$

(8) A uniform and straight plank rests with its middle point upon a rough horizontal cylinder, their directions being perpen-

dicular to each other; to find the greatest weight which can be suspended from one end of the plank without its sliding off the cylinder.

Let W be the weight of the plank, and P the attached weight; r the radius of the cylinder, $2a$ the length of the plank, $\tan\lambda$ the coefficient of friction. Then P will be given by the relation

$$\frac{P}{W}=\frac{r\lambda}{a-r\lambda}.$$

(9) A uniform rod rests over a smooth peg, its lower end being supported by a rough horizontal plane: to find its position of equilibrium when bordering upon motion.

If $2a=$ the length of the rod, $h=$ the height of the peg above the horizontal plane, and $\tan\epsilon=$ the coefficient of friction: then θ, the inclination of the rod to the horizon in the required position, is determined by the equation

$$a\sin 2\theta.\sin(\theta+\epsilon)=2h\sin\epsilon.$$

(10) A uniform beam AB, (fig. 48), of which the end B presses against a rough vertical plane CD, is supported by a fine string AC attached to a fixed point C in the plane; to find the position of the beam when bordering upon motion.

Let the point B be on the point of ascending; let $\mu=$ the coefficient of friction, a = the length of the beam, $CA=l$, $\angle ACB=\theta$, $\angle ABD=\phi$. Then θ may be found from the equation

$$(4a^2-4l^2-\mu^2l^2)\tan^2\theta-2\mu l^2\tan\theta+4a^2-l^2=0;$$

and then ϕ may be determined by the equation

$$a\sin\phi=l\sin\theta.$$

If B be on the point of sliding downwards, μ must be replaced by $-\mu$.

(11) A uniform rod rests within a rough circle, the plane of which is vertical: to investigate the position of the rod when the friction can only just maintain the equilibrium.

If α denote the angle between the rod and the radius through

either extremity, $\tan \epsilon$ the coefficient of friction, and θ the inclination of the rod to the horizon in the required position,

$$\tan \theta = \frac{\sin 2\epsilon}{\cos 2\epsilon - \cos 2\alpha}.$$

(12) A square board $ABCD$, (fig. 49), the plane of which is vertical, rests with its side AD in contact with a rough vertical wall, which is perpendicular to the plane of the board; the side AB resting, at a point indefinitely near to B, upon a rough peg: to find the least value of the coefficient of friction, supposing it to be the same for the wall and for the peg.

The least value of the coefficient of friction is equal to $\sqrt{(2)} - 1$.

(13) An elliptical cylinder, placed between a smooth vertical plane and a rough horizontal one, with the major axis of the ellipse inclined at an angle of 45° to the horizon, is just prevented by friction from sliding; to find the coefficient of friction.

If e be the eccentricity of the ellipse, the coefficient of friction will be equal to $\frac{1}{2}e^2$.

(14) A homogeneous solid hemisphere is capable of rolling on its curve surface upon a horizontal plane, the friction being such as to prevent all sliding; to find the moment of a couple which may keep it at rest with its base inclined at an angle of 30° to the horizon.

If W be the weight and a the radius of the hemisphere, the moment of the couple will be equal to $\frac{3}{16} Wa$.

(15) A sphere of radius a is just supported on a rough plane, inclined at an angle of 45° to the horizon, by a weightless rod, the lower extremity of which is attached by a hinge to the inclined plane, and the higher to the surface of the sphere, at a point where the radius is parallel to the plane; the rod and the centre of the sphere lying in a vertical plane which cuts the inclined plane at right angles. To find the length of the rod, the coefficient of friction being equal to $\tan \epsilon$.

The length of the rod is equal to $a \operatorname{cosec} \epsilon$.

(16) A straight uniform beam is placed upon two rough planes, of which the inclinations to the horizon are α and α', and the coefficients of friction $\tan\lambda$ and $\tan\lambda'$; to find the limiting value of the angle of inclination of the beam to the horizon at which it will rest, and the relation between the weight of the beam and each of the two normal pressures upon the planes.

Let θ be the required limiting angle; R, R', the normal pressures on the planes; and W the weight of the beam. Then

$$2\tan\theta = \cot(\alpha'+\lambda') - \cot(\alpha-\lambda),$$

$$\frac{R}{\cos\lambda\sin(\alpha'+\lambda')} = \frac{W}{\sin(\alpha-\lambda+\alpha'+\lambda')} = \frac{R'}{\cos\lambda'\sin(\alpha-\lambda)}.$$

(17) A right cone is placed on its base upon a rough inclined plane, the inclination of which is gradually increased: to investigate the condition that a motion of rolling and of sliding may take place simultaneously.

If $\frac{1}{2}\beta$ denote the angle of indifference, and α the vertical angle of the cone, the required condition is expressed by the equation

$$\tan\frac{\beta-\alpha}{2} = \frac{3\sin\alpha}{5-3\cos\alpha}.$$

(18) A uniform rectangular plank AB, (fig. 50), of given weight W, is just supported against a rough vertical wall BC by a weight P suspended at one end of a string which passes through a ring at O, vertically above B, and of which the other end is tied to A. To find the least value of the normal pressure on the wall, and the corresponding magnitude of P.

If $\tan\epsilon$ denote the coefficient of friction, the least value of the normal pressure is $\frac{1}{2}W\cot\epsilon$, and the corresponding magnitude of P is $\frac{1}{2}W\operatorname{cosec}\epsilon$.

(19) When a person tries to pull out a two-handled drawer by pulling one of the handles in a direction perpendicular to its front, to find the condition under which the drawer will stick fast.

The drawer will stick fast, whatever be the force employed, if the coefficient of friction be not less than the ratio of the length of either side of the drawer to the distance between its handles.

CHAPTER IV.

EQUILIBRIUM OF SEVERAL BODIES.

If there be a system of bodies mutually acting on each other by contact, by connecting rods, or in any conceivable way, it will be necessary, in the determination of the circumstances of equilibrium, to represent the unknown actions and reactions by appropriate symbols. We shall then have to write down the equations of equilibrium for each body separately, including among the known forces to which it is subject, the unknown actions which it experiences from its connection with the other bodies of the system. From these different sets of equations, taken conjointly, we shall have to determine the circumstances of equilibrium.

Sect. 1. *No Friction.*

(1) AB (fig. 51) is a uniform beam, capable of motion about its middle point D; CE is a beam, moveable about a hinge C in the vertical line through D, and pressing against the beam AB from the extremity B of which a weight P is suspended; to determine the positions of the beams for equilibrium, having given that CD is equal to AD or BD.

Let $AD = CD = BD = a$, $\angle ACD = \theta$; $GC = b$, G being the centre of gravity of the beam CE; $R =$ the action and reaction of the two beams at A; $W =$ the weight of the beam CE. Then for the equilibrium of CE, taking moments about C, we have

$$R.2a\cos\theta = W.b\sin\theta;$$

and for the equilibrium of AB, taking moments about D,

$$R.a\cos\theta = P.a\sin 2\theta, \quad \text{or } R = 2P\sin\theta;$$

from these two equations, by the elimination of R, we get

$$Wb\sin\theta = 2Pa\sin 2\theta = 4Pa\sin\theta\cos\theta,$$

and therefore $\theta = 0$, or $\cos\theta = \dfrac{bW}{4aP}$;

which determine the required positions of the beams.

(2) Two spheres O and O', (fig. 52), rest upon two smooth inclined planes AC and AC', and press against each other; to determine their position.

Let W, W', be the weights of the spheres O, O'; R their mutual action and reaction; α, α', the inclinations of the planes AC, AC', to the horizon; θ the inclination of the line OO', joining the centres of the spheres, to the horizon.

Then for the equilibrium of the sphere O, resolving forces parallel to AC,

$$R\cos(\alpha+\theta) = W\sin\alpha;$$

and for the equilibrium of the sphere O', resolving forces parallel to AC',

$$R\cos(\alpha'-\theta) = W'\sin\alpha'.$$

Eliminating R between these two equations,

$$W\sin\alpha\cos(\alpha'-\theta) = W'\sin\alpha'\cos(\alpha+\theta),$$

$$W\tan\alpha\,(1+\tan\alpha'\tan\theta) = W'\tan\alpha'\,(1-\tan\alpha\tan\theta),$$

and therefore $\tan\theta = \dfrac{W'\tan\alpha' - W\tan\alpha}{(W'+W)\tan\alpha'\tan\alpha}$.

(3) Three spheres O, O', O'', (fig. 53), are placed in contact within a hollow sphere; a vertical plane through the centre of the hollow sphere being supposed to contain the centres of the three solid spheres; to find their positions of equilibrium.

Let C be the centre of the hollow sphere; O, O', O'', the centres of the solid spheres; join OC, $O'C$, $O''C$; let W, W', W'', be the weights of the three spheres; $CO = r$, $CO' = r'$, $CO'' = r''$; $\angle OCO' = \alpha$, $\angle O'CO'' = \alpha''$; $\theta =$ the inclination of $O'C$ to the horizon.

Then, since the actions of the hollow sphere on the solid ones all three pass through the point C, we have for the equilibrium of the solid spheres, taking moments about C, observing that, if each of the spheres be in equilibrium, they would likewise be at rest if rigidly connected together as a single body,

$$Wr\cos(\theta - \alpha) + W'r'\cos\theta + W''r''\cos(\theta + \alpha'') = 0,$$

$$Wr(\cos\alpha + \sin\alpha\tan\theta) + W'r' + W''r''(\cos\alpha'' - \sin\alpha''\tan\theta) = 0;$$

and therefore $$\tan\theta = \frac{W''r''\cos\alpha'' + W'r' + Wr\cos\alpha}{W''r''\sin\alpha'' - Wr\sin\alpha}.$$

(4) A hollow cylinder stands upon a horizontal plane, and a rigid imponderable rod, in a vertical plane through the axis of the cylinder, passes over the upper edge of the cylinder and rests against its interior surface: a given weight is attached to the other extremity of the rod, and the cylinder, which is prevented from slipping by a small obstacle on the plane, is just on the point of turning over. To determine the weight of the cylinder.

Let a = the length of the rod AEB, (fig. 54), $AE = x$; c = the diameter of the cylinder and W = its weight; P = the weight suspended from B, θ = the inclination of AB to the horizon; and let R, S, denote the reactions of the cylinder against the rod.

For the equilibrium of the rod we have, resolving horizontally,

$$R\cos\theta = P\sin\theta \qquad (1),$$

and, taking moments about E,

$$Rx\sin\theta = P(a - x)\cos\theta,$$

or, since $$x\cos\theta = c,$$

$$Rc\sin\theta = P(a\cos\theta - c)\cos\theta \qquad (2).$$

From (1) and (2),

$$c\sin^2\theta = (a\cos\theta - c)\cos^2\theta,$$

$$\cos\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}} \qquad (3):$$

hence $$(a - x)\cos\theta = c^{\frac{1}{3}}(a^{\frac{2}{3}} - c^{\frac{2}{3}}).$$

For the equilibrium of the cylinder and rod, regarded as one system, taking moments about O, we have

$$W.\tfrac{1}{2}c = P(a - x)\cos\theta = Pc^{\frac{1}{3}}(a^{\frac{2}{3}} - c^{\frac{2}{3}}),$$

$$W = 2P.\frac{a^{\frac{2}{3}} - c^{\frac{2}{3}}}{c^{\frac{2}{3}}}.$$

COR. From (1) and (3),

$$R = P\left(\frac{a^{\frac{2}{3}} - c^{\frac{2}{3}}}{c^{\frac{2}{3}}}\right)^{\frac{1}{2}},$$

and, resolving vertically for the equilibrium of the rod,

$$P = S\cos\theta,$$

$$S = P\left(\frac{a}{c}\right)^{\frac{1}{3}}.$$

(5) A sphere and cone of given weights are placed in contact on two inclined planes, the intersection of which is a horizontal line; to determine the circumstances of equilibrium.

Let W, W', be the weights of the sphere and the cone, which we may suppose to be applied at their centres of gravity G, G', (fig. 55). Let R be the action of the plane AB upon the sphere, and S the mutual action of the sphere and cone: if ϕ denote the semiangle of the cone, then evidently the line of action of S will make an angle ϕ with the plane AB'. The plane AB' will exert at right angles to itself an action upon every element of the base of the cone; the resultant of all these actions will be some force R' applied at some point E of the base of the cone in the line AB'. Let α, α', be the inclinations of the two planes to the horizon.

For the equilibrium of the sphere we have, resolving forces parallel to the plane AB,

$$W\sin\alpha = S\cos(\alpha + \alpha' - \phi)\,\ldots\ldots\ldots\ldots\ldots(1),$$

and, resolving forces at right angles to the plane,

$$R = W\cos\alpha + S\sin(\alpha + \alpha' - \phi)\,\ldots\ldots\ldots\ldots(2);$$

the equation of moments is an identical equation, since all the forces which act upon the sphere pass through its centre.

Again, for the equilibrium of the cone, resolving the forces which act upon it parallel to the plane AB'.

$$W'\sin\alpha' = S\cos\phi\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

resolving forces at right angles to the plane AB',

$$R' = W'\cos\alpha' + S\sin\phi\,\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and taking moments about G', the lines EH, mG', being represented by x, y,

$$R'x = Sy\cos\phi\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

From the equations (1) and (3),

$$\frac{W \sin \alpha}{W' \sin \alpha'} = \frac{\cos(\alpha + \alpha' - \phi)}{\cos \phi} \text{.................}(6),$$

from which $\tan\phi$ may be readily determined: this relation is the only condition to which the cone and sphere are subject to secure equilibrium; as will be evident when it is observed that the three equations (2), (4), (5), introduce four unknown quantities R, R', x, y, each of the three equations at least one, which are not involved in (1) and (3). From this it is evident that there will be an infinite number of positions of equilibrium, or that if ϕ only have the value given by (6), the cone and sphere will rest in contact in whatever manner they may be placed on the two planes, and whatever be their magnitudes.

The values of ϕ being determined by (6), S will be determined by (1) or (3), and therefore R, R', from (2), (4), respectively. Then from the equation (5) we may determine x, provided that y be given; and y can be given only by our knowing the magnitudes of the cone and sphere, and the particular position of equilibrium in which we may choose to place them.

(6) Two uniform rods AC, $A'C$, of which the lower extremities are situated in the same horizontal plane, and prevented from sliding, lean against each other at the point C, and are in equilibrium; to determine the relation between their angles of inclination to the horizon, the small area of mutual contact at C being vertical.

Let W, W', be the weights of the rods AC, $A'C$, respectively, and ϕ, ϕ', their angles of inclination to the horizon; then

$$W \cot\phi = W' \cot\phi'.$$

Franchini; *Memorie della Societa Italiana*, Tom. XVI. P. I. p. 237; 1813.

(7) An inextensible string binds tightly together two smooth cylinders of given radii; to find the ratio of the mutual pressure between the cylinders to the tension by which it is produced.

If R be the mutual pressure, T the tension of the string, r, r', the radii of the cylinders; then

$$\frac{R}{T} = \frac{4\,(rr')^{\frac{1}{2}}}{r+r'}.$$

(8) A sphere of given weight and radius is suspended by a string of given length from a fixed point, to which point also is attached another given weight by a string so long that the weight hangs below the sphere; to find the angle which the string, to which the sphere is attached, makes with the vertical.

If P denote the weight, Q the weight and sphere together, a the radius of the sphere, and b the distance of its centre from the point of suspension; then the required angle will be equal to

$$\sin^{-1}\left(\frac{Pa}{Qb}\right).$$

(9) A heavy sphere is placed upon three spheres, each equal to itself, which rest in contact on a horizontal plane: to find the pressure on each, and also the horizontal force which must be applied to each to preserve the equilibrium.

If W = the weight of each sphere, R = the pressure on each, and F = the required horizontal force; then

$$R = \frac{W}{\sqrt{6}}, \quad F = \frac{W}{3\sqrt{2}}.$$

(10) A sphere, of which C is the centre, is attached to a point O by a fine string and touches a uniform rod OB moveable in a vertical plane about a hinge at O: to find the position of equilibrium.

Let W = the weight of the sphere, W' = the weight of the rod, r = the radius of the sphere, $2a$ = the length of the rod, b = the distance between O and C, and θ = the inclination of OC to the vertical: then

$$\cot\theta = \frac{Wb^2}{W'ar} + \left(\frac{b^2}{r^2} - 1\right)^{\frac{1}{2}}.$$

(11) A rod AB (fig. 56) is fixed at a given angle of inclination to the vertical; a rod CD is attached to AB by connections

at the points B, C, a weight W being suspended from the extremity D; to determine the pressures exerted by AB upon CD, the weight of CD being neglected.

Let F, G, denote the resolved parts of the pressures at B, C, on CD, estimated along its length; and R, S, the pressures at right angles to the former; let $CD=b$, $CB=c$; then, α being the inclination of the rods to the vertical,

$$R=\frac{b}{c}\,W\sin\alpha,\qquad S=\frac{b-c}{c}\,W\sin\alpha,$$

$$F+G=W\cos\alpha,$$

the single value of F or G being indeterminate.

(12) A uniform rod OA, moveable about a smooth hinge at O, rests tangentially against a smooth sphere, of which C is the centre, and which is placed upon a smooth horizontal plane passing through O: the sphere is tied to O by a string. To find the tension of the string.

If $a=$ the length of the rod, $W=$ its weight, $r=$ the radius of the sphere, $c=$ the distance of C from O, and $T=$ the tension of the string,

$$T=W.\frac{ar}{c^3}.\frac{c^2-2r^2}{(c^2-r^2)^{\frac{1}{2}}}.$$

(13) A beam AB (fig. 57) is moveable in a vertical plane about its middle point G: another beam, hanging by a string, attached to its higher end, from a point in the same plane, rests with its lower end C upon GB. To determine the position of a point E in AG at which a given weight W must be suspended so as to preserve equilibrium.

If $W=$ the weight of AB, and $P=$ that of the other beam, then

$$GE=\frac{P}{2W}.\,CG.$$

(14) Two spheres A, B, (fig. 58), of equal weights and volumes, support a third sphere C, the weight of which is equal to that of

A or B; the spheres A, B, being attached by equal strings to a fixed point O: to find the condition of equilibrium.

If α denote the inclination of either string, and β of either AC or BC to the vertical,

$$\tan\beta = 3\tan\alpha.$$

(15) Two equal uniform rods, each equally inclined to the horizon, support a sphere which rests against their higher extremities: the lower ends of the rods are fixed to hinges in a horizontal line: to find the inclination of either rod to the horizon.

If $2a$ = the length and W = the weight of each rod, r = the radius and W' = the weight of the sphere, and $2c$ = the distance between the two hinges, then θ, the required angle, is determined by the equation

$$(c - 2a\cos\theta)^2 . \{(W + W')^2\cos^2\theta + W'^2\sin^2\theta\}$$
$$= r^2 (W + W')^2 . \cos^2\theta.$$

(16) Two equal uniform rods AOB, $A'OB'$, (fig. 59), in a vertical plane, are connected together by a smooth hinge at their middle point O: their lower ends B, B', rest on a smooth horizontal plane, and their upper ends A, A', are tied together by a fine string: a sphere C is placed between them: to find the tension of the string.

If r denote the radius and W the weight of the sphere; $2a$ the length and W the weight of each rod; α the inclination of each rod to the vertical, and T the tension of the string; then

$$T = \frac{Wr\cos\alpha + (2P + W)\,a\sin^2\alpha}{2a\sin^2\alpha\cos\alpha}.$$

(17) Two equal balls, (fig. 60), are placed within a hollow vertical cylinder, open at both ends, which rests upon a horizontal plane: the weight of each ball is W and radius r, the radius of the cylinder being r': to find the least value of the weight of the cylinder in order that it may not be upset by the balls.

If W' = the least weight,

$$W' = 2W\left(1 - \frac{r}{r'}\right).$$

(18) A paraboloid of revolution is placed with its vertex downwards and its axis vertical, between two planes equally inclined to the horizon; to find the greatest ratio which the length of the paraboloid may have to its latus rectum, so that, if the solid be divided by a plane through its axis and the line of intersection of the inclined planes, the two parts may remain in equilibrium.

Let $\alpha =$ the inclination of either plane to the vertical, $h =$ the greatest length of the axis of the paraboloid, and $l =$ its latus rectum; then

$$\left(\frac{h}{l}\right)^{\frac{1}{2}} = \frac{15\pi}{64} \cdot \frac{1 + \sin^2 \alpha}{\sin^3 \alpha} \cdot \cos \alpha.$$

SECT. 2. *Friction.*

(1) Two equal uniform beams AK, AK', (fig. 61), which are capable of revolving in a vertical plane about a point A to which their lower extremities are attached, have their upper extremities connected by a string KK'; a heavy sphere is placed between the two beams; supposing the string to contract, to determine its tension when the sphere is just going to be forced upwards, the friction between the sphere and each of the beams being given.

It is plain that the two beams must make equal angles with the vertical line AL which passes through A, because the centre of gravity of the system consisting of the two beams and the sphere must lie in this line.

Let R, R', denote the actions of the beams upon the sphere at right angles to their lengths, and F, F', their actions along their lengths which are due to roughness. Let 2α be the angle at which the two beams are inclined to each other, T the tension of the string KK'; W the weight of the sphere, W' of each of the beams, and $2a$ the length of each.

Then for the equilibrium of the sphere we have, resolving forces parallel to LA,

$$(F + F') \cos \alpha + W = (R + R') \sin \alpha \ldots\ldots\ldots\ldots (1);$$

resolving at right angles to LA,

$$(F' - F) \sin \alpha = (R - R') \cos \alpha \quad\text{................ (2)};$$

and taking moments about O, the centre of the sphere,

$$F \,.\, OE = F' \,.\, OE', \quad \text{or } F = F' \quad\text{................ (3)}.$$

From (2) and (3) we have

$$R' = R \quad\text{................................ (4)}.$$

Now supposing the sphere to be on the point of being disturbed by the contraction of the string, one or both of the points E, E', of the sphere must be on the point of sliding along the corresponding beams. Suppose that sliding is on the point of taking place at E.

Then, μ being the coefficient of friction between the sphere and the beam AK, we have

$$F = \mu R;$$

and therefore from (1), (3), (4),

$$2\mu R \cos \alpha + W = 2R \sin \alpha,$$

and therefore, putting $\mu = \tan \epsilon$,

$$R = \frac{W}{2(\sin \alpha - \mu \cos \alpha)} = \frac{W \cos \epsilon}{2 \sin (\alpha - \epsilon)} \quad\text{.............. (5)}.$$

Also, from (3) and (4),

$$\frac{F'}{R'} = \frac{F}{R} = \mu,$$

and therefore $\quad F' = \mu R'.$

Hence we see that, if μ' be the coefficient of friction between the sphere and the beam AK', μ is not greater than μ', since the greatest value of F' will be $\mu' R'$. If μ be less than μ', the sphere would, with the slightest increase in the tension of KK', begin to roll along AK' without sliding; and, if μ be equal to μ', the sphere would begin to slide at both points simultaneously.

Again, for the equilibrium of AK we have, taking moments about A, it being remembered that the actions and reactions between the sphere and the beams are equal and opposite,

$$R \,.\, AE + W' \,.\, a \sin \alpha = T \,.\, 2a \cos \alpha;$$

and therefore, r being the radius of the sphere,

$$Rr \cot \alpha + W'a \sin \alpha = 2Ta \cos \alpha;$$

hence, putting for R its value given in (5),

$$\frac{Wr \cos \epsilon \cos \alpha}{2 \sin \alpha \sin (\alpha - \epsilon)} + W'a \sin \alpha = 2Ta \cos \alpha,$$

and therefore

$$T = \frac{Wr \cos \epsilon}{4a \sin \alpha \sin (\alpha - \epsilon)} + \tfrac{1}{2} W' \tan \alpha.$$

(2) AB (fig. 51) is a uniform beam, capable of motion about its middle point D; CE is a beam, moveable about a hinge C in the vertical line through D, and pressing against the beam AB, from the extremity B of which a weight P is suspended; CD, AD, BD, are equal lines; from observing the magnitude of the angle ACD when the end A of the beam AB is on the point of sliding in the direction CE, to find the coefficient of friction between the two beams.

Let G be the centre of gravity of the beam CE; μ the coefficient of friction; R the mutual action of the two beams at right angles to CE; $\angle ACD = \beta = \angle CAD$; $AD = a = BD$; $CG = b$, $4Q$ the weight of the beam CE.

Then for the equilibrium of CE we have, taking moments about C,

$$R \,.\, 2a \cos \beta = 4Q \,.\, b \sin \beta,$$

or

$$aR \cos \beta = 2bQ \sin \beta \;\ldots\ldots\ldots\ldots\ldots\; (1);$$

and for the equilibrium of AB, taking moments about D,

$$R \,.\, a \cos \beta = \mu R \,.\, a \sin \beta + P \,.\, a \sin 2\beta,$$

$$R (\cos \beta - \mu \sin \beta) = P \sin 2\beta \;\ldots\ldots\ldots\ldots\; (2).$$

From (1) and (2) there is

$$\frac{2bQ \sin \beta}{a \cos \beta} (\cos \beta - \mu \sin \beta) = 2P \sin \beta \cos \beta;$$

and therefore $\quad bQ (1 - \mu \tan \beta) = aP \cos \beta,$

$$\mu = \frac{bQ - aP \cos \beta}{bQ \tan \beta}.$$

(3) A weight W (fig. 62) is suspended from the middle point of a rigid rod without weight, connecting the centres O, O', of two equal heavy wheels, which rest on a rough inclined plane; the wheel O is locked: to find the greatest inclination of the plane which is consistent with the equilibrium of the carriage.

Let P be the weight, and r the radius of each of the wheels; let $OO' = 2a$, $\phi =$ the inclination of the plane to the horizon; let R, R', be the reactions of the plane on the wheels at right angles to itself; μR the friction on the wheel O, μ being the coefficient of friction; F the action of the plane on the wheel O' at right angles to R'; X, Y, the resolved parts, parallel and perpendicular to the plane, of the action of the wheel O' on the rod OO'; and X', Y', the similarly resolved parts of the reaction.

For the equilibrium of the wheel O and the rod OO', regarded as one system, we have, resolving forces parallel to the inclined plane,

$$\mu R = X + (P + W) \sin \phi \quad\dots\dots\dots\dots\dots (1);$$

resolving forces at right angles to the plane,

$$R + Y = (P + W) \cos \phi \quad\dots\dots\dots\dots\dots (2);$$

and, taking moments about O,

$$\mu Rr + 2aY = Wa \cos \phi \quad\dots\dots\dots\dots\dots (3).$$

Again, for the equilibrium of the wheel O', we have, taking moments about the point of contact of this wheel with the plane,

$$X'r = Pr \sin \phi, \quad \text{or } X' = P \sin \phi \quad\dots\dots\dots\dots\dots (4).$$

From the equations (1) and (4), observing that X' is by the nature of action and reaction equal to X, we get

$$\mu R = (2P + W) \sin \phi \quad\dots\dots\dots\dots\dots (5).$$

Again, from (2) and (3),

$$\mu r R + 2a (P + W) \cos \phi - 2aR = Wa \cos \phi,$$

$$(2a - \mu r) R = a (2P + W) \cos \phi \quad\dots\dots\dots\dots\dots (6).$$

From (5) and (6) we obtain for the required inclination of the plane,

$$\tan \phi = \frac{\mu a}{2a - \mu r}.$$

COR. Having ascertained ϕ, we know R from (5) and X' or X from (4), and therefore Y from (2); also F being the only force acting on the wheel O' which does not pass through its centre, it is evident that F must be equal to zero.

(4) Two equal beams AC, BC, are connected by a smooth hinge at C, and are placed in a vertical plane with their lower extremities A and B resting on a rough horizontal plane; from observing the greatest value of the angle ACB for which equilibrium is possible, to determine the coefficient of friction at the ends A and B.

If β be the greatest value of $\angle ACB$, and μ be the coefficient of friction at each of the ends; then

$$\mu = \tfrac{1}{2} \tan \tfrac{1}{2}\beta.$$

SECT. 3. *Systems of Beams.*

(1) Two uniform rods AC, BC, (fig. 63), are connected together by a smooth hinge-joint at C, their other ends being fastened to two smooth fixed hinges A, B, in a vertical line: to find the magnitudes and directions of the pressures on the hinges and of the mutual action of the rods at the joint.

It is frequently convenient in problems of this class, to make use of diagrams in which the several members of the system are represented to the eye in a state of slight detachment; the actions and reactions being indicated by arrowed lines not running into each other. The student will thereby escape falling into errors of sign in writing down the equations of equilibrium, to which he is liable from confounding together actions and reactions. In fact, the problem thereby resolves itself into the consideration of the equilibrium of several distinct bodies.

Let $AC = 2a$, $BC = 2b$, and let $\tan \alpha$, $\tan \beta$, be represented by m, n, respectively. The horizontal and vertical components of the actions and reactions on the rods are indicated in the diagram, as well as the weights of the rods.

For the equilibrium of AC there is, resolving horizontally,

$$X + X'' = 0 \quad \text{.......................... (1)},$$

vertically, $$Y + Y'' = P \qquad (2),$$

and, taking moments about C,

$$X \cdot 2a \cos\alpha + Y \cdot 2a \sin\alpha = P \cdot a \sin\alpha,$$

or, $$2X + 2mY = mP \qquad (3).$$

In like manner, for the equilibrium of BC,

$$X' = X'' \qquad (4),$$

$$Y' = Q + Y'' \qquad (5),$$

and $$Y' \cdot 2b \sin\beta = X' \cdot 2b \cos\beta + Q \cdot b \sin\beta,$$

or, $$2nY' = 2X' + nQ \qquad (6).$$

From (1), (2), (3), there is

$$2X'' + 2mY'' = mP \qquad (7).$$

From (4), (5), (6), there is

$$2X'' - 2nY'' = nQ \qquad (8).$$

From (7) and (8) we have

$$Y'' = \tfrac{1}{2} \cdot \frac{mP - nQ}{m + n} \qquad (9).$$

Also, from (7) and (8), we have

$$X'' = \tfrac{1}{2} \cdot \frac{mn}{m+n} \cdot (P + Q) \qquad (10).$$

Hence also, by (1) and (4),

$$X = -\tfrac{1}{2} \frac{mn}{m+n} (P + Q) \qquad (11),$$

$$X' = \tfrac{1}{2} \frac{mn}{m+n} (P + Q) \qquad (12).$$

From (2) and (9),

$$Y = \frac{mP + n(2P + Q)}{2(m+n)} \qquad (13).$$

From (5) and (9)

$$Y' = \frac{nQ + (2Q + P)m}{2(m+n)} \qquad (14).$$

The two components of the pressures exerted at A, C, B, upon each rod having been ascertained, the required directions and magnitudes of these pressures are therefore known.

(2) At the middle points of the sides of any polygon $ABCDE$...... (fig. 64), and at right angles to them, are applied a series of forces P, Q, R, ..., respectively proportional to the sides; the sides of the polygon are perfectly rigid, and capable of moving freely about the angular points $A, B, C, D, \ldots$; to determine the form of the polygon that it may be in equilibrium, the lengths of the sides being given.

Let p, q, r, s, ... denote the mutual actions of the sides of the polygon at the angles A, B, C, D, ..., of which the directions will lie in certain straight lines $bB\beta$, $cC\gamma$, $dD\delta$, ...

For the equilibrium of the side BC we have, resolving forces at right angles to it,

$$Q = q \sin \angle CB\beta + r \sin \angle BCc \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

resolving forces parallel to BC,

$$q \cos \angle CB\beta = r \cos \angle BCc \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and, taking moments about the middle point of BC,

$$q \sin \angle CB\beta = r \sin \angle BCc \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Dividing (3) by (2), we have

$$\tan \angle CB\beta = \tan \angle BCc,$$

and therefore, $\quad \angle CB\beta = \angle BCc \ldots\ldots\ldots\ldots\ldots\ldots (4)$;

hence also, from (2) or (3), $\quad q = r \ldots\ldots\ldots\ldots\ldots\ldots (5)$.

Again, from (1) and (3), we have

$$Q = 2r \sin \angle BCc;$$

in precisely the same manner we may find that

$$R = 2r \sin \angle DC\gamma,$$

and therefore $\quad \dfrac{Q}{R} = \dfrac{\sin \angle BCc}{\sin \angle DC\gamma}$;

but, by the hypothesis,

$$\frac{Q}{R} = \frac{BC}{DC} = \frac{\sin \angle BDC}{\sin \angle CBD};$$

hence $\quad \dfrac{\sin \angle BCc}{\sin \angle DC\gamma} = \dfrac{\sin \angle BDC}{\sin \angle CBD}$;

but from the geometry it is evident that

$$\angle BCc + \angle DC\gamma = \angle BDC + \angle CBD;$$

hence we readily see that

$$\angle BCc = \angle BDC \text{ .... } (6).$$

In just the same way we might prove that

$$\angle CB\beta = \angle BAC,$$

and therefore, by (4),

$$\angle BDC = \angle BAC \text{ .... } (7).$$

From this relation (7) it is plain that a circle passing through the three points A, B, C, must pass likewise through the point D; similarly we might shew that this circle, since it passes through B, C, D, must likewise pass through E, and so on indefinitely; hence we see that when the sides of the polygon are arranged consistently with equilibrium, all its angular points must be situated in the circumference of a single circle.

From (5) we gather that

$$p = q = r = s = \text{......},$$

or that the mutual pressures at all the angular points are equal. It is evident also from the relation (6), that all the lines $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, ... are tangents to the circle passing through A, B, C, D, ...

The value of the mutual pressure at each of the angular points is easily obtained: thus, as we have shewn,

$$Q = 2r \sin \angle BCc;$$

but since $\angle BCc$ is equal to half the angle subtended by BC at the centre of the circle circumscribing the polygon, it is clear that

$$\sin \angle BCc = \frac{\frac{1}{2}BC}{\text{radius}};$$

hence $$r = \text{radius} \times \frac{Q}{BC},$$

and therefore, $$p = q = r = s \text{ ......} = k\rho,$$

where ρ denotes the radius and k the ratio between any one of the forces and the corresponding side of the polygon.

Fuss; *Mémoires de St. Pétersb.* 1817, 1818, p. 46.

The following is a different solution of the same problem:— Let the forces P, Q, R, ... be represented in magnitude by the lines $2AB$, $2BC$, $2CD$, ..., to which they are proportional. Instead of the force $2AB$ acting at the middle point of the side AB, apply two forces, each equal to AB, one at the end A and the other at the end B of the side AB; each of these forces being at right angles to the side AB. Again, instead of the force $2BC$ acting at the middle point of BC, apply a force BC at C, and a force BC at the extremity B of the side AB, (which we are at liberty to do, because the point B of AB is rigidly attached to the point B of BC,) each of these forces being at right angles to BC. Now, according to this distribution of the forces, the only force which could twist BC about C, is the action of the rod AB upon the end B of BC; and therefore for the equilibrium of BC it is necessary that this action should take place exactly along BC. Hence conversely the action of CB upon BA will take place entirely in the direction CB. Let this action be denoted by R.

Thus, the line AB is acted upon at the point B by a force AB at right angles to AB, a force BC at right angles to BC, and a force R in the direction CB: but, by the principle of the parallelogram of forces, the forces AB and BC at B are equivalent to a single force AC acting at right angles to AC; hence for the equilibrium of AB we have, taking moments about A,

$$R \,.\, AB \,.\, \sin \angle ABC = AC \,.\, AB \cos \angle BAC,$$

or,
$$R \sin \angle ABC = AC \cos \angle BAC.$$

Similarly, for the equilibrium of the side CD,

$$R \sin \angle BCD = BD \cos \angle BDC;$$

and therefore
$$\frac{\sin \angle ABC}{\sin \angle BCD} = \frac{AC}{BD} \frac{\cos \angle BAC}{\cos \angle BDC}.$$

But, by the geometry,

$$\frac{\sin \angle BAC}{\sin \angle BDC} = \frac{\dfrac{BC}{AC} \sin \angle ABC}{\dfrac{BC}{BD} \sin \angle BCD} = \frac{BD}{AC} \frac{\sin \angle ABC}{\sin \angle BCD}.$$

Hence from these two relations we have

$$\frac{\sin \angle BAC}{\sin \angle BDC} = \frac{\cos \angle BAC}{\cos \angle BDC},$$

$$\tan \angle BAC = \tan \angle BDC, \quad \angle BAC = \angle BDC;$$

which shews, as in the former solution, that the sides of the polygon must be so arranged that its angular points may all lie in the circumference of a single circle.

(3) A quadrilateral $ABCD$, (fig. 65), consists of four rigid rods, which are capable of free motion about the angular points A, B, C, D; supposing the points A, C, and B, D, to be attached together by strings AC and BD in given states of tension, to determine the geometrical conditions necessary for the equilibrium of the quadrilateral.

Let P, Q, represent the tensions of the strings AC, BD. Let K, L, M, N, denote the actions and reactions between the four pairs of points (A, B), (B, C), (C, D), (D, A).

The force P acting upon the point A in the direction AC, is equivalent to a force, in the direction AB,

$$= P\frac{\sin CAD}{\sin BAD} = P\frac{\sin ADB}{\sin BAD}\cdot\frac{DO}{AO} = P\frac{OD\,.\,AB}{BD\,.\,OA};$$

and to some force (F suppose) in AD.

Similarly, the force Q acting upon the point B in the direction BD, is equivalent to

$$\text{a force, in } BA, \; = Q\frac{OC\,.\,AB}{AC\,.\,OB},$$

and some force (G suppose) in BC.

Hence clearly the point A is solicited by a force $F - N$ in AD, and a force

$$P\frac{OD\,.\,AB}{BD\,.\,OA} - K \text{ in } AB \,\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and therefore for its equilibrium we have

$$F - N = 0, \text{ and } P\frac{OD\,.\,AB}{BD\,.\,OA} - K = 0.$$

Similarly for the equilibrium of the point B there is

$$G - L = 0, \text{ and } Q\frac{OC.AB}{AC.OB} - K = 0 \text{..........} (2).$$

From (1) and (2) we have

$$P\frac{OD.AB}{BD.OA} = Q\frac{OC.AB}{AC.OB},$$

and therefore

$$\frac{P.OD}{BD.OA} = \frac{Q.OC}{AC.OB},$$

which is the condition for the equilibrium of the quadrilateral.

Euler; *Act. Acad. Petrop.* 1779, P. II. p. 106.

The following is a different solution of the same problem:

For the equilibrium of the rod AB there is, taking moments about B,

$$N.BD.\sin\angle BDA = P.BO.\sin\angle BOC;$$

and for the equilibrium of the rod CD, taking moments about C,

$$N.CA.\sin\angle CAD = Q.CO.\sin\angle BOC;$$

hence obviously

$$\frac{BD\sin\angle ODA}{CA\sin\angle OAD} \text{ or } \frac{BD.AO}{AC.DO} = \frac{P.BO}{Q.CO}.$$

(4) Four rigid rods AB, BC, CD, DA, (fig. 66), are so joined together that they are capable of revolving freely about the angular points of the quadrilateral which they form; these rods are attached together, two and two, viz. those which are contiguous, by strings $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, in given states of tension; to determine the form of the quadrilateral which shall correspond to the equilibrium of the rods.

Let A, B, C, D, denote the tensions of the strings $a\alpha$, $b\beta$, $c\gamma$, $d\delta$. Then the force A in $a\alpha$ upon the point a is equivalent to a force, in BA,

$$= A\frac{\sin \alpha a D}{\sin A a D} = A\frac{\sin a D\alpha . \frac{D\alpha}{a\alpha}}{\sin AD\alpha . \frac{DA}{A\alpha}} = A\frac{A\alpha . D\alpha}{a\alpha . DA};$$

and to a force, in aD,

$$= A\frac{\sin Aa\alpha}{\sin AaD} = A\frac{\sin aA\alpha \, . \, \frac{A\alpha}{a\alpha}}{\sin aAD \, . \, \frac{AD}{aD}} = A\frac{A\alpha \, . \, D\alpha}{a\alpha \, . \, DA}$$

$$= A' \text{ suppose.}$$

But the force A' in aD is equivalent to

$$\text{a force in } AD, \; = A'\frac{\sin aDB}{\sin ADB} = A'\frac{\sin aBD \, . \, \frac{Ba}{Da}}{\sin ABD \, . \, \frac{BA}{DA}}$$

$$= A'\frac{Ba \, . \, DA}{Da \, . \, BA} = A\frac{A\alpha \, . \, Ba}{a\alpha \, . \, BA};$$

and to a force in BD, $= A'\dfrac{\sin ADa}{\sin ADB}$

$$= A'\frac{\sin DAa \, . \, \frac{Aa}{Da}}{\sin DAB \, . \, \frac{AB}{DB}} = A'\frac{Aa \, . \, BD}{AB \, . \, Da} = A\frac{A\alpha \, . \, Aa \, . \, BD}{a\alpha \, . \, DA \, . \, BA}.$$

Thus we see that the force A, acting upon the point a in the direction $a\alpha$, is equivalent to the three forces

$$A\frac{aD \, . \, A\alpha}{AD \, . \, a\alpha} \text{ in } BA \text{ upon } A, \quad A\frac{A\alpha \, . \, aB}{AB \, . \, a\alpha} \text{ in } AD \text{ upon } A,$$

and

$$\frac{aA \, . \, A\alpha \, . \, BD}{AB \, . \, AD \, . \, a\alpha} \text{ in } BD \text{ upon } B.$$

Similarly, the force A acting upon the point α in the direction αa, is equivalent to

$$A\frac{\alpha B \, . \, Aa}{AB \, . \, a\alpha} \text{ in } DA \text{ upon } A, \quad A\frac{Aa \, . \, \alpha D}{AD \, . \, a\alpha} \text{ in } AB \text{ upon } A,$$

and

$$A\frac{\alpha A \, . \, Aa \, . \, DB}{AD \, . \, AB \, . \, a\alpha} \text{ in } DB \text{ upon } D.$$

Now these three forces are equal and opposite to the three former, and therefore the string $a\alpha$ with a tension A produces

the same effect, and may therefore be replaced by a string BD with a tension

$$A\frac{aA\,.\,Aa\,.\,BD}{AB\,.\,AD\,.\,a\alpha}.$$

In the same way we may shew, that the tension of $c\gamma$ is equivalent to a string BD, of which the tension is equal to

$$C\frac{cC\,.\,C\gamma\,.\,BD}{CB\,.\,CD\,.\,c\gamma}.$$

Hence the tensions of $a\alpha$, $c\gamma$, together, are equivalent to a string BD with a tension

$$A\frac{Aa\,.\,A\alpha\,.\,BD}{BA\,.\,DA\,.\,a\alpha}+C\frac{Cc\,.\,C\gamma\,.\,BD}{BC\,.\,DC\,.\,c\gamma}.$$

Similarly it may be shewn, that the tensions $b\beta$, $d\delta$, are equivalent to a string AC with a tension

$$B\frac{Bb\,.\,B\beta\,.\,CA}{AB\,.\,CB\,.\,b\beta}+D\frac{D\delta\ \ Dd\,.\,AC}{AD\,.\,CD\,.\,d\delta}.$$

Hence, by the result of the preceding problem, the condition of equilibrium is expressed by the relation

$$\frac{OB\,.\,OD}{BD^2}\left(\frac{B\,.\,Bb\,.\,B\beta}{AB\,.\,CB\,.\,b\beta}+\frac{D\,.\,D\delta\,.\,Dd}{AD\,.\,CD\,.\,d\delta}\right)$$
$$=\frac{OA\,.\,OC}{AC^2}\left(\frac{A\,.\,Aa\,.\,A\alpha}{BA\,.\,DA\,.\,a\alpha}+\frac{C\,.\,Cc\,.\,C\gamma}{BC\,.\,DC\,.\,c\gamma}\right).$$

Euler; *Act. Acad. Petrop.* 1779, P. 2, p. 106.

(5) Two equal uniform beams AB, AC, moveable about a hinge at A, are placed upon the convex circumference of a circle in a vertical plane; to find their inclination to each other when they are in their position of equilibrium.

Let $2a$ = the length of each beam, 2θ = their inclination to each other, and r = the radius of the circle. Then θ will be determined by the equation

$$r\cos\theta = a\sin^3\theta.$$

(6) A and C (fig. 67) in the same vertical line are fixed points, about which beams AB, CD, are freely moveable by

hinge joints; AB is supported in a horizontal position by CD, with which it is connected by a hinge joint at D, and has a weight suspended at B: to find the pressure at C, the weights of the beams being neglected.

Let H and V be the horizontal and vertical pressures at C, and P the weight suspended from B. Then

$$H = P.\frac{AB}{AC}, \quad V = P.\frac{AB}{AD},$$

and therefore the whole pressure at C is equal to

$$P.AB.\left(\frac{1}{AC^2} + \frac{1}{AD^2}\right)^{\frac{1}{2}}.$$

(7) Two uniform rods AC, BD, (fig. 68), are connected together by a hinge at D. Their ends A, B, resting on a smooth horizontal plane, are tied together by a string. To find the tension of the string.

If $AC = 2a$, $BD = 2b$, $AB = c$, $\angle CAB = \alpha$, $\angle DBA = \beta$, and P, Q, denote the weights of AC, BD, respectively, the tension of the string will be equal to

$$\frac{Pa \sin\alpha + Qb \sin\beta}{c \tan\alpha \tan\beta}.$$

(8) Three uniform beams AB, BC, CD, of the same thickness, and of lengths l, $2l$, l, respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius of which is equal to $2l$, so that the middle point of BC and the extremities of AB, CD, are in contact with the sphere; to compare the pressure at the middle point of BC, and the pressures at A and D, with the weight of the three beams.

Let W be the weight of the three beams taken together; R the pressure at each of the points A and D; and R' the pressure at the middle point of BC. Then

$$\frac{R}{W} = \frac{3}{40}, \quad \frac{R'}{W} = \frac{91}{100}.$$

(9) Four equal uniform beams AB, BC, CD, DE, (fig. 69), connected together by joints at their extremities, rest in equi-

librium in a vertical plane; the distances AE and CF, of which the latter is perpendicular to AE and vertical, are given; to determine the conditions of equilibrium.

If α, β, be the inclinations of AB and ED, BC and DC, to the horizon; we must have

$$\tan\alpha = 3\tan\beta.$$

Draw BK at right angles to AE; let $CF=a$, $AF=b$, $FK=x$, $BK=y$; then from the equation in α and β, and the geometry of the figure, we may get

$$x=\frac{a^2+2b^2-(a^4+a^2b^2+b^4)^{\frac{1}{2}}}{2b}, \quad y=\frac{2a^2+b^2-(a^4+a^2b^2+b^4)^{\frac{1}{2}}}{2a}.$$

These values of x and y are obtained by Couplet in his *Recherches sur la Construction des Combles de Charpente*, in the *Mémoires de l'Académie des Sciences de Paris*, 1731, p. 69.

CHAPTER V.

EQUILIBRIUM OF FLEXIBLE STRINGS.

THE form of equilibrium assumed by a uniform flexible string sustained at its two extremities and acted on by gravity, attracted the attention of Galileo[1], who, from a want of sufficient examination, concluded it to be a parabola; this mistake may have arisen from the fact, that in the immediate neighbourhood of its lowest point it approximates very nearly to the parabolic form. The inaccuracy of Galileo's conclusion was experimentally ascertained by Joachim Jungius[2]. This subject having been at last successfully investigated by James Bernoulli[3], he proposed the problem of the chaînette, the name which he gave to the required curve, as a trial of skill to the mathematicians of the day. The four mathematicians who succeeded in arriving at correct solutions of the problem were, James Bernoulli, by whom it had been proposed, his brother John, Leibnitz, and Huyghens: their four solutions appeared without analysis in the *Acta Eruditorum* for the year 1691, Jun. pp. 273—282. A demonstration of the results of these four illustrious mathematicians was first published by David Gregory, in the *Philosophical Transactions* for the year 1697.

The form of equilibrium of the chaînette or catenary, of which the thickness is supposed to be uniform, having been thoroughly discussed, James Bernoulli[4] next directed his attention to more complicated problems of the same character; he investigated the form of equilibrium when the thickness varies

[1] *Mechanica;* Dialogo 2, p. 131.
[2] *Geometria Empyrica.*
[3] *Acta Eruditorum*, Lips. 1690, Mai. p. 217; *Opera*, Tom. I. p. 424.
[4] *Acta Eruditorum*, Lips. 1691, Jun. p. 289; *Opera*, Tom. I. p. 449.

from point to point according to any assigned law, and, conversely, determined the law of its variation that the string may hang in assigned curves: he likewise considered the problem of the catenary when the string is extensible, the extension of each element being assumed according to the law established experimentally by Hooke[1] to vary as the tension. The analysis of these problems, of which the solutions only were published by James Bernoulli, was supplied by John Bernoulli[2]. The consideration of the general conditions of the equilibrium of flexible strings was first attempted by Hermann[3], whose investigations, however, were not free from error; a more accurate analysis was furnished by John Bernoulli[4], who has particularly examined various cases of the equilibrium of strings acted on by central forces.

Among the numerous mathematicians who afterwards discussed the theory of the equilibrium of flexible strings, may be mentioned Euler[5], Clairaut[6], Krafft[7], Legendre[8], Fuss[9], Venturoli[10], and Poisson[11].

SECT. 1. *Free Inextensible String; general Conditions of Equilibrium.*

To investigate the conditions for the equilibrium of an inextensible string, of which the density and thickness vary from point to point according to any assigned law; the accelerating forces which act upon the string being any whatever.

[1] *De Potentia Restitutiva, or Spring.*
[2] *Lectiones Mathematicæ in usum Hospitalii, Opera*, Tom. IV. p. 387.
[3] *Phoronomia*, lib. I. cap. 3, and Append. § V.
[4] *Opera*, Tom. IV. p. 234.
[5] *Comment. Petrop.* Tom. III.; *Nov. Comment. Petrop.* Tom. XV. and Tom. XX.
[6] *Miscellanea Berolinensia*, Tom. VII. p. 270, 1743.
[7] *Nov. Comment. Petrop.* Tom. V. p. 143; 1754 and 1755.
[8] *Mém. Acad. Par.* 1786, p. 20.
[9] *Nova Acta Petrop.* Tom. XII. p. 145, 1794.
[10] *Elements of Mechanics*, by Cresswell, Part I. p. 62.
[11] *Traité de Mécanique*, Tom. I. p. 564. seconde édition.

Let APB (fig. 70) be any portion of the string in a position of rest; Pp being a small element of its length; x, y, z, and $x+\delta x$, $y+\delta y$, $z+\delta z$, the co-ordinates of P and p respectively; s the length of the string reckoned from some assigned point up to P, and $s+\delta s$ the length up to p; t the tension of the string at P.

The resolved parts, parallel to the axes of x, y, z, of the force exerted upon the point P of the element Pp by the portion AP of the string, will evidently be

$$-t\frac{dx}{ds}, \quad -t\frac{dy}{ds}, \quad -t\frac{dz}{ds};$$

and therefore, since each of these three forces must be some function of s, it is plain by Taylor's theorem that the resolved parts of the force exerted on the element Pp by the portion pB of the string, will be

$$t\frac{dx}{ds}+\frac{d}{ds}\left(t\frac{dx}{ds}\right)\delta s,$$

$$t\frac{dy}{ds}+\frac{d}{ds}\left(t\frac{dy}{ds}\right)\delta s,$$

$$t\frac{dz}{ds}+\frac{d}{ds}\left(t\frac{dz}{ds}\right)\delta s.$$

Again, let X, Y, Z, be the sums of the resolved parts of the accelerating forces which act upon the element Pp; ρ the density of the string at P, and k the area of a section at right angles to its length at that point. Then clearly the mass of the portion Pp of the string will be $k\rho\delta s$, which therefore for a constant value of δs will vary as $k\rho$; hence evidently the product $k\rho$, which we will call m, may be taken to measure the *massiveness* of the string at the point P; it will be convenient to call it *the unit of mass* at the point P. The impressed moving force then of the element Pp, will have for its resolved parts parallel to the co-ordinate axes,

$$mX\delta s, \quad mY\delta s, \quad mZ\delta s.$$

Hence clearly for the equilibrium of Pp we must have, equating to zero the sum of the resolved forces which act upon

it parallel to each of the three axes, and dividing the three resulting equations by δs,

$$\left.\begin{aligned} \frac{d}{ds}\left(t\frac{dx}{ds}\right) + mX = 0, \\ \frac{d}{ds}\left(t\frac{dy}{ds}\right) + mY = 0, \\ \frac{d}{ds}\left(t\frac{dz}{ds}\right) + mZ = 0, \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (a);$$

which three equations constitute the conditions of equilibrium of the entire string.

By the elimination of t we readily obtain the three following equations,

$$dy\int mZds = dz\int mYds,$$

$$dz\int mXds = dx\int mZds;$$

$$dx\int mYds = dy\int mXds,$$

any two of which will be differential equations to the required curve of equilibrium.

COR. 1. From the equations (a) we have also

$$t\frac{dx}{ds} = -\int mXds, \quad t\frac{dy}{ds} = -\int mYds, \quad t\frac{dz}{ds} = -\int mZds;$$

squaring and adding these equations, and observing that

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1 \dots\dots\dots\dots\dots\dots\dots\dots (b),$$

we obtain for the value of the tension at any point,

$$t^2 = \left(\int mXds\right)^2 + \left(\int mYds\right)^2 + \left(\int mZds\right)^2.$$

We may obtain also another expression for the tension: differentiating (b) with respect to s, we get

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \dots\dots\dots\dots\dots (c);$$

hence, multiplying the three equations (*a*) by dx, dy, dz, in order, and adding the resulting equations, we have, by the aid of (*b*) and (*c*),

$$t = C - \int m\,(Xdx + Ydy + Zdz),$$

where C is an arbitrary constant.

Cor. 2. If the whole string lie entirely within one plane, let the plane of xy be so chosen as to coincide with this plane; then the three differential equations to the string will be reduced to the single one

$$dx\int mYds = dy\int mXds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d);$$

and the two formulæ for the tension will become

$$t^2 = \left(\int mXds\right)^2 + \left(\int mYds\right)^2,$$

$$t = C - \int m\,(Xdx + Ydy).$$

These two formulæ for the tension, and also the differential equation (*d*) to the string, coincide with those given by Fuss; *Mémoires de St. Pétersbourg*, 1794, pp. 150, 151.

Sect. 2. *Parallel Forces.*

(1) A flexible string fixed at any two points A and B, (fig. 71), is acted on by gravity; supposing the unit of mass to vary according to any assigned law as we pass from one point to another, to find the equation to the catenary of rest; and conversely, the curve being known, to determine the law of the unit of mass.

Let the axis of y extend vertically upwards, and let the axis of x be horizontal, the plane xOy coinciding with the plane which contains the catenary. Then, since

$$X = 0, \quad Y = -g,$$

we have, by the first two of the equations (*a*) of section (1),

$$\frac{d}{ds}\left(t\frac{dx}{ds}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$\frac{d}{ds}\left(t\frac{dy}{ds}\right)=mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Integrating the equation (*a*), we get

$$t\frac{dx}{ds}=C,$$

where C is a constant quantity: let τ denote the tension at the lowest point of the curve, then evidently $\tau = C$, and therefore

$$t\frac{dx}{ds}=\tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

From (*b*) and (*c*), we have

$$\tau\frac{d}{ds}\frac{dy}{dx}=mg,$$

and therefore $$\tau\frac{dy}{dx}=\int mg\,ds;$$

but evidently at the lowest point of the catenary $\frac{dy}{dx}=0$, and therefore, supposing a to be the value of s at the lowest point,

$$0=\int^{a} mg\,ds;$$

hence $$\tau\frac{dy}{dx}=g\int_{a}^{s} m\,ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

If m be given in terms of the variables x, y, s, the form of the catenary may be determined from (*d*).

Again, differentiating (*d*), we obtain

$$m=\frac{\tau}{g}\frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (e),$$

a formula by which m may be computed for every point of the string when the form of the catenary is given. Also from (*c*) we get

$$t = \tau \frac{ds}{dx} \dots\dots\dots\dots\dots\dots\dots\dots (f),$$

which gives the tension at any point of the catenary when its form is known.

John Bernoulli; *Lectiones Mathematicæ*,
Lect. 38, 39, 40; Opera, Tom. III.

(2) A flexible string AOB, (fig. 72), fixed at two points A and B, is acted on by gravity; the unit of mass at any point P varies inversely as the square root of the length OP measured from the lowest point O; to find the equation to the catenary.

Let the origin of co-ordinates be taken at O, x being horizontal, and y vertical, and the plane of xy coinciding with the plane of the catenary; also let O be the origin of s.

Then, if μ be the unit of mass at the end of a length c from the lowest point,

$$m = \mu \frac{c^{\frac{1}{2}}}{s^{\frac{1}{2}}},$$

and therefore by $(1, d)$, α being in the present case zero, we have

$$\tau \frac{dy}{dx} = g\mu c^{\frac{1}{2}} \int_0^s \frac{ds}{s^{\frac{1}{2}}} = 2g\mu c^{\frac{1}{2}} s^{\frac{1}{2}};$$

hence, putting for the sake of brevity

$$\frac{2g\mu c^{\frac{1}{2}}}{\tau} = \frac{1}{\beta^{\frac{1}{2}}},$$

we get
$$\frac{dy}{dx} = \left(\frac{s}{\beta}\right)^{\frac{1}{2}}, \quad \frac{dy^2}{dx^2} = \frac{s}{\beta},$$

$$\beta \frac{d}{dx} \frac{dy^2}{dx^2} = \frac{ds}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}},$$

$$\frac{\beta \dfrac{d}{dx} \dfrac{dy^2}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}} = 1;$$

integrating with respect to x we obtain

$$2\beta \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = x + C;$$

but $x=0$, $\dfrac{dy}{dx}=0$, simultaneously; hence $C=2\beta$, and therefore

$$2\beta\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}=x+2\beta \quad\ldots\ldots\ldots\ldots\ldots\ldots (a);$$

squaring and transposing,

$$4\beta^2\frac{dy^2}{dx^2}=(x+2\beta)^2-4\beta^2,$$

$$2\beta\, dy=\{(x+2\beta)^2-4\beta^2\}^{\frac{1}{2}}dx;$$

integrating we have

$$C+2\beta y=\tfrac{1}{2}(x+2\beta)(x^2+4\beta x)^{\frac{1}{2}}-2\beta^2\log\{x+2\beta+(x^2+4\beta x)^{\frac{1}{2}}\};$$

but $x=0$, $y=0$, simultaneously; hence

$$C=-2\beta^2\log(2\beta);$$

hence, eliminating C,

$$2\beta y=\tfrac{1}{2}(x+2\beta)(x^2+4\beta x)^{\frac{1}{2}}-2\beta^2\log\frac{x+2\beta+(x^2+4\beta x)^{\frac{1}{2}}}{2\beta},$$

which is the required equation to the catenary.

Cor. From (a) we get

$$\frac{ds}{dx}=\frac{x+2\beta}{2\beta},$$

and therefore, by $(1, f)$,

$$t=\tau\frac{ds}{dx}=\frac{\tau}{2\beta}(x+2\beta),$$

which gives the tension at any point of the curve.

John Bernoulli; *Lect. Math.*, Opera, Tom. III. p. 497.

(3) To find the law of variation of the unit of mass of a catenary acted on by gravity that it may hang in the form of a semicircle with its diameter horizontal.

The notation remaining the same as in (2), the equation to the catenary will be

$$x^2=2ay-y^2,$$

where a denotes the radius of the semicircle: hence

$$a^2-x^2=(a-y)^2, \qquad y=a-(a^2-x^2)^{\frac{1}{2}};$$

$$\frac{dy}{dx}=\frac{x}{(a^2-x^2)^{\frac{1}{2}}},\qquad \frac{d^2y}{dx^2}=\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}};$$

also $$\frac{ds^2}{dx^2}=1+\frac{dy^2}{dx^2}=\frac{a^2}{a^2-x^2},\qquad \frac{ds}{dx}=\frac{a}{(a^2-x^2)^{\frac{1}{2}}};$$

and therefore, by (1, e),

$$m=\frac{\tau}{g}\frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}}=\frac{\tau}{g}\frac{a}{a^2-x^2}=\frac{\tau a}{g(a-y)^2};$$

or the unit of mass at any point varies inversely as the square of its depth below the horizontal diameter of the semicircle.

COR. By (1, f) we have for the tension at any point

$$t=\tau\frac{ds}{dx}=\frac{\tau a}{(a^2-x^2)^{\frac{1}{2}}}=\frac{\tau a}{a-y}.$$

John Bernoulli; *Opera*, Tom. III. p. 502.

(4) To find the length of a uniform chain ALB, (fig. 73), suspended from two points A and B in the same horizontal line, when the stress on each point of support is equal to the whole weight of the chain; to find also the depth of the lowest point L of the chain below the line AB, and the direction of its tangent at A or B.

Let $yCLO$ be vertical, OL being equal to a length of the chain of which the weight is equal to the tension of the lowest point L, Ox horizontal; PM at right angles to Ox. $OM=x$, $PM=y$, $OL=c$, $ALB=l$, $AC=BC=a$.

Then the equation to the curve will be

$$y=\tfrac{1}{2}c\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and also $$l=c\left(\epsilon^{\frac{a}{c}}-\epsilon^{-\frac{a}{c}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let m denote the unit of mass of the chain, which will be the same at all its points; then the tension at P will be equal to

$$mgy=\tfrac{1}{2}mcg\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right),$$

and therefore at B to $\tfrac{1}{2}mcg\left(\epsilon^{\frac{a}{c}}+\epsilon^{-\frac{a}{c}}\right)$;

but by the hypothesis the tension at B is equal to mgl, and therefore by (2) to

$$mcg\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}});$$

hence $$\tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) = mcg\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}});$$

$$\tfrac{1}{2}\epsilon^{\frac{a}{c}} = \tfrac{3}{2}\epsilon^{-\frac{a}{c}},$$

$$\epsilon^{\frac{2a}{c}} = 3, \quad \frac{2a}{c} = \log_e 3, \quad \frac{a}{c} = \tfrac{1}{2}\log_e 3 \ldots\ldots\ldots(3).$$

Hence from (2) we have

$$l = \frac{2a}{\log_e 3}\left(3^{\frac{1}{2}} - \frac{1}{3^{\frac{1}{2}}}\right) = \frac{4a}{3^{\frac{1}{2}}\log_e 3},$$

which gives the length of the chain.

Again, putting $x = a$, we have from (1),

$$OC = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}),$$

and therefore $$CL = (\tfrac{1}{2}\epsilon^{\frac{a}{c}} + \tfrac{1}{2}\epsilon^{-\frac{a}{c}} - 1)\,c$$

$$= \left(\frac{3^{\frac{1}{2}} + 3^{-\frac{1}{2}}}{2} - 1\right)\frac{2a}{\log_e 3}, \text{ from (3)},$$

$$= \left(\frac{2}{3^{\frac{1}{2}}} - 1\right)\frac{2a}{\log_e 3} = \frac{2a}{\log_e 3}\,\frac{2 - 3^{\frac{1}{2}}}{3^{\frac{1}{2}}},$$

which gives the depth of the lowest point of the chain below the line AB.

Again, from (1) we have

$$\frac{dy}{dx} = \tfrac{1}{2}\,(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}),$$

and therefore, ϕ denoting the inclination of the chain at B to the horizon,

$$\tan\phi = \tfrac{1}{2}\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}) = \tfrac{1}{2}\left(3^{\frac{1}{2}} - \frac{1}{3^{\frac{1}{2}}}\right) = \frac{1}{3^{\frac{1}{2}}};$$

hence $$\phi = \frac{\pi}{6}.$$

(5) A uniform string $A'ALBB'$ (fig. 74) is placed over two supports A and B in the same horizontal line, so as to remain in equilibrium; having given the length of the string, and the distance of the points of support, to find the pressure which they have to bear.

Let L be the lowest point of the curve ALB, OLy a vertical line through L, where OL is equal to a length of the chain, the weight of which is equal to the tension at L; Ox horizontal. Then, Ox, Oy, being taken as the axes of co-ordinates, we shall have for the equation to the curve ALB, putting $OL = c$,

$$y = \tfrac{1}{2}c\,(\epsilon^{\frac{s}{c}} + \epsilon^{-\frac{s}{c}}) \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, if m be the unit of mass at each point of the string, the tension at P will be equal to

$$mgy \text{ or } \tfrac{1}{2}mcg\,(\epsilon^{\frac{s}{c}} + \epsilon^{-\frac{s}{c}});$$

hence, if $AC = BC = a$, the tension at B will be equal to

$$\tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But the tension at B is evidently equal to the weight of BB', and therefore, if $BB' = s$, to the expression mgs; hence

$$mgs = \tfrac{1}{2}mcg\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}),$$

or

$$s = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Suppose that the length of the whole string $A'ALBB'$ is $2l$; then the length of the portion LBB' will be l, and $l - s$ will be the length of BL. Hence, by the nature of the catenary,

$$l - s = \tfrac{1}{2}c\,(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}) \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Adding together the equations (3) and (4), we obtain

$$l = c\epsilon^{\frac{a}{c}},$$

whence c is made to depend upon the known quantities a and l: hence the expression (2) for the tension at B is known.

Differentiating (1), we get

$$\frac{dy}{dx} = \tfrac{1}{2}\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right):$$

but, if $LP = s'$,

$$s' = \tfrac{1}{2}c\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right), \quad \frac{ds'}{dx} = \tfrac{1}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right);$$

hence evidently
$$\frac{dy}{ds'} = \frac{\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}}{\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}},$$

and therefore, if ϕ denote the angle between the line BB' and the curve BL at B,

$$\cos\phi = \frac{\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}}{\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}} \quad\quad (5).$$

Let P denote the pressure on the point B, and τ the tension of the string at B; then

$$P^2 = 2\tau^2 + 2\tau^2 \cos\phi$$
$$= 2\tau^2 (1 + \cos\phi);$$

and therefore, from (2) and (5),

$$P^2 = \tfrac{1}{2}m^2c^2g^2\left(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}\right)^2\left\{1 + \frac{\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}}{\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}}\right\}$$
$$= m^2c^2g^2\left(\epsilon^{\frac{a}{c}} + \epsilon^{-\frac{a}{c}}\right)\epsilon^{\frac{a}{c}}$$
$$= m^2c^2g^2\left(1 + \epsilon^{\frac{2a}{c}}\right)$$
$$P = mcg\left(1 + \epsilon^{\frac{2a}{c}}\right)^{\frac{1}{2}},$$

which gives the required value of the pressure, c having been previously determined.

(6) A uniform chain ABC (fig. 75) is suspended from a point A above an inclined plane RS: having given the angle which the chain at the point of suspension and which the plane makes with the horizon, and also the length of the whole chain, to find the length of the portion BC which is in contact with the plane.

Let $ABLA'$ denote the catenary, of which AB is an arc, L being the lowest point. Let P be any point in the curve AL; ϕ the inclination of the curve at P to the horizon, t the tension at B; α, β, the values of ϕ at A, B, respectively; c the length of chain of which the weight is equal to the tension at L; m the unit of mass of the chain; $LP=s$, $ABC=l$, $BC=l'$.

Then, by the nature of the catenary,

$$t \cos\beta = mcg \qquad (1),$$

$$s = c \tan\phi \qquad (2).$$

Now it is evident that the tension at B is equal to $mgl' \sin\beta$; hence, from (1),

$$mgl' \sin\beta \cos\beta = mcg, \qquad c = l' \sin\beta \cos\beta \qquad (3).$$

Again, from (2), we have

$$LBA = c \tan\alpha, \qquad LB = c \tan\beta,$$

and therefore $$l - l' = c(\tan\alpha - \tan\beta);$$

hence, from (3),

$$l - l' = l' \sin\beta \cos\beta (\tan\alpha - \tan\beta),$$

$$l \cos\alpha = l' (\cos\alpha + \sin\alpha \sin\beta \cos\beta - \sin^2\beta \cos\alpha)$$

$$= l' \cos\beta \,.\, \cos(\alpha - \beta)$$

$$l' = \frac{l \cos\alpha}{\cos\beta \cos(\alpha-\beta)}.$$

(7) AOB (fig. 72) is a flexible string acted on by gravity, and is in a position of rest; the unit of mass at any point varies as the cosine of the angle at which an element of the curve at the point is inclined to the horizon; to find the equation to the catenary.

Assuming $m = \beta \dfrac{dx}{ds}$, where β is some constant quantity, the equation to the catenary will be

$$x^2 = \frac{2\tau}{\beta g} y;$$

which shews that the catenary is the common parabola.

James Bernoulli; *Act. Erudit.* Lips. Jun.; *Opera*, Tom. I. p. 449. John Bernoulli; *Opera*, Tom. III. p. 501.

(8) To find the equation to the catenary when the unit of mass varies as $x \cos \phi$, where ϕ is the angle of inclination of the element of the curve at any point to the horizon.

Assuming $m = \beta x \dfrac{dx}{ds}$, the required equation will be

$$6\tau y = g\beta x^3,$$

which belongs to a cubical parabola.

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(9) To find the equation to the catenary when the unit of mass varies as $x^{\frac{1}{2}} \cos \phi$.

Assuming $m = \beta x^{\frac{1}{2}} \dfrac{dx}{ds}$, the equation will be

$$16 g^2 \beta^2 x^5 = 225 \tau^2 y^2.$$

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(10) To find the equation to the catenary when the unit of mass varies as $y^n \sin \phi$, where n is any positive quantity.

If the origin of co-ordinates be so chosen that the axis of x passes through the lowest point of the catenary, and that $y = \infty$ when $x = 0$, the required equation will be

$$xy^n = -\frac{(n+1)\,\tau}{ng\beta}.$$

James Bernoulli; *Ib.* John Bernoulli; *Ib.*

(11) To find the law of the variation of the unit of mass when the catenary is the common parabola.

The construction and notation being the same as in (2),

$$m = \frac{2\tau}{g\,(a^2 + 4x^2)^{\frac{1}{2}}},$$

a being the latus rectum of the parabola.

John Bernoulli; *Opera*, Tom. III. p. 504.

(12) A chain suspended at its extremities from two tacks in the same horizontal line, forms itself into a cycloid; to find the unit of mass at any point of the string and the weight of the arc between this and the lowest point.

Let w denote the weight of the arc; then, taking the ordinary equations to the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta),$$

we shall have

$$m = \frac{\tau (\sec \tfrac{1}{2}\theta)^2}{4ag}, \quad w = \tau \tan \tfrac{1}{2}\theta.$$

(13) One end of a heavy chain is attached to a fixed point A, and the other to a weight which is placed on a rough horizontal plane passing through A, and the chain hangs through a slit in the horizontal plane; to find the greatest distance of the weight from A, at which equilibrium is possible.

If a be the length of the chain, x the greatest distance of the weight from A, μ the coefficient of friction, and n twice the ratio between the given weight and that of the chain,

$$\epsilon^{\frac{a}{x}} = \left[\frac{1 + \{1 + \mu^2 (1+n)^2\}^{\frac{1}{2}}}{\mu(1+n)}\right]^{\mu(1+n)}.$$

(14) A uniform chain is suspended from two tacks in the same horizontal line at a distance $2a$ from each other; to determine the length of the chain that the stress on the tacks may be a minimum.

Let c denote a length of the chain of which the weight is equal to the tension at the lowest point; and let l denote the required length of the chain. Then

$$\epsilon^{\frac{a}{c}} = \left\{\frac{\frac{a}{c} + 1}{\frac{a}{c} - 1}\right\}^{\frac{1}{2}}, \quad l = c\left(\epsilon^{\frac{a}{c}} - \epsilon^{-\frac{a}{c}}\right);$$

from the former equation $\frac{a}{c}$ and therefore c is to be determined, and then l will be given by the latter.

If for instance $2a = 10$ feet, then $c = 4.168$ feet nearly, and $l = 12.578$ feet nearly.

Diarian Repository, p. 644.

(15) A chain acted on by gravity hangs in the form of a curve, of which $a^3y = x^4$ is the equation; to find where the unit of mass is a maximum, and its maximum value.

When m is a maximum, x and y being the co-ordinates of the point,

$$x = \frac{a}{2^{\frac{1}{2}}}, \quad y = \tfrac{1}{4}a, \quad m = \frac{2 \,.\, 3^{\frac{1}{2}}\tau}{ag}.$$

The law of the mass of the chain is erroneously investigated in the *Lady's and Gentleman's Diary* for the year 1745; see also *Diarian Repository*, p. 435.

(16) A uniform chain of length $2l$ is suspended from two points in a horizontal line, the distance $2a$ between which is given: to investigate an equation for the determination of the inclination of the curve to the horizon at either point of support.

If ϕ represent the required angle,

$$a = l \cot \phi \log \left(\tan \frac{\pi + 2\phi}{4}\right).$$

(17) A chain $ABCDE$, (fig. 76), passes over a smooth pulley B, the portions BA, DE, of the chain hanging freely, while the portion CD rests upon a smooth table. Supposing CD to be half the length of the whole chain, to compare the length of the chain with the height of B above CD.

The required ratio is equal to

$$2\,(3 + \sqrt{3}).$$

(18) A perfectly flexible chain of variable thickness hangs in the form of a curve defined by the equation

$$\frac{y}{b} = \log\left(\sec \frac{x}{b}\right),$$

the axis of x being horizontal: to find the law of variation of the thickness and of the tension.

The tension of the string at any point varies as the area of the section of the string, each varying as $\sec\left(\frac{x}{b}\right)$.

(19) A uniform chain of length l hangs over two fixed points, which are in a horizontal line: from its middle point is suspended by one end another chain of equal thickness and of length l'. Supposing each of the two tangents of the former chain at its middle point to make an angle θ with the vertical, to find the distance between the two fixed points, and to shew that θ can never exceed a certain value.

The distance between the two fixed points is equal to

$$l' \tan\theta \,.\, \log\left\{\frac{l+l'}{l'} \cdot \frac{\tan\frac{\theta}{2}}{\tan\theta}\right\}.$$

The value of θ can never exceed that given by the equation

$$\tan^2\frac{\theta}{2} = \frac{l-l'}{l+l'}.$$

SECT. 3. *Central Forces.*

(1) To find the equation to a flexible string in a position of equilibrium under the action of any central attractive force.

Let APB (fig. 77) be any portion of the string; S the centre of force; P any point in the string, PT a tangent at this point; SY a perpendicular from S upon PT; p a point of the string indefinitely near to P, and pk a tangent at p. Also let OP, Op, be the normals at P, p, O being therefore the centre and OP the radius of curvature at P; let OP produced meet pk in k.

Let $OP = \rho$, $SP = r$, $SY = p$, $\angle\, SPT = \phi$, $\angle\, kOp = \psi$, $m =$ the unit of mass at P; $t =$ the tension at P and $t + dt$ at p; $Pp = ds$, $F =$ the central force at P.

Then for the equilibrium of the element Pp we have, resolving the forces which act upon it at right angles to PT,

$$Fmds \sin\phi = (t + dt)\cos pkO = (t + dt)\sin\psi,$$

or, retaining infinitesimals of the first order,

$$Fmds \sin\phi = t\psi = t\,\frac{ds}{\rho};$$

and therefore

$$Fm \sin\phi = \frac{t}{\rho} \text{.............................} (a);$$

and resolving forces parallel to PT we have

$$Fmds \cos\phi = (t + dt) \sin Okp - t$$
$$= (t + dt) \cos\psi - t,$$

or, retaining infinitesimals of the first order only,

$$Fmds \cos\phi = dt;$$

and therefore, $ds \cos\phi$ being equal to dr,

$$Fmdr = dt \text{..............................} (b).$$

From the equation (a), since

$$\rho = -r\frac{dr}{dp} \text{ and } \sin\phi = \frac{p}{r},$$

we have

$$Fmdr + \frac{dp}{p}t = 0;$$

and therefore from (b)

$$\frac{dp}{p} + \frac{dt}{t} = 0,$$
$$\log(pt) = \log C,$$

where C is an arbitrary constant; and therefore

$$p = \frac{C}{t} = \frac{C}{\int Fmdr}, \text{........................} (c),$$

which is the equation to the catenary in p and r when the form of F is known.

Let θ be the angle between SP and any fixed line; then

$$p = \frac{r^2\,d\theta}{(dr^2 + r^2 d\theta^2)^{\frac{1}{2}}},$$

and therefore from (c), putting $\int Fmdr = R$,

$$Rr^2\,d\theta = C\,(dr^2 + r^2\,d\theta^2)^{\frac{1}{2}},$$
$$R^2r^4\,d\theta^2 = C^2\,(dr^2 + r^2\,d\theta^2),$$

and therefore

$$d\theta = \frac{Cdr}{r\,(R^2r^2 - C^2)^{\frac{1}{2}}} \text{.....................} (d),$$

the differential equation to the catenary between r and θ. This is the form in which the solution is given by John Bernoulli[1].

The value of the tension at any point of the catenary is given by (b), when the expression for F in terms of r is known.

The relations at which we have arrived may be deduced from the general equations of equilibrium of section (1); the method however of the tangential and normal resolution is more convenient in the case of central forces.

If the central force be repulsive instead of attractive, we must replace F by $-F$, wherever it occurs in the above formulæ.

(2) To find the form of the catenary when the central force is attractive and varies inversely as the square of the distance; the unit of mass being invariable.

Let AOB (fig. 78) be the catenary; S the centre of force; SO the radius vector which meets the curve at right angles.

$$\tau = \text{the tension at } O, \text{ and } SO = c.$$

Then, if k denote the attraction at the distance c,

$$R = \int F m dr = \int k \frac{c^2}{r^2} m dr = C' - \frac{mkc^2}{r},$$

where C' is an arbitrary constant: but, by (1, b), $t = R$, and therefore

$$\tau = C' - mkc;$$

hence $$t = R = \tau + mkc - \frac{mkc^2}{r} \quad \text{.......................} \quad (a).$$

Hence, from (1, d), we have

$$d\theta = \frac{C dr}{r\left\{\left(\tau + mkc - \dfrac{mkc^2}{r}\right)^2 r^2 - C^2\right\}^{\frac{1}{2}}};$$

but from (1, c), since $p = c$ and $t = \tau$ at the point O, we see that $C = c\tau$; therefore

$$d\theta = \frac{c\tau dr}{r\left\{\left(\tau + mkc - \dfrac{mkc^2}{r}\right)^2 r^2 - c^2\tau^2\right\}^{\frac{1}{2}}}.$$

[1] *Opera*, Tom. IV. p. 238.

For the sake of simplicity put $\tau = nmkc$; then

$$d\theta = \frac{ncdr}{r\left\{\left(n+1-\frac{c}{r}\right)^2 r^2 - n^2c^2\right\}^{\frac{1}{2}}}$$

$$= \frac{ncdr}{r\left\{(n+1)^2r^2 - 2(n+1)cr + c^2 - n^2c^2\right\}^{\frac{1}{2}}};$$

the equation to the catenary resulting from the integration of this differential equation will be of three different forms according as n is greater than, equal to, or less than unity.

First, suppose that n is greater than unity; then the integral of the equation will be, supposing that $\theta = 0$ when $r = c$,

$$r = \frac{(n-1)c}{n\cos\left\{\left(n^2-1\right)^{\frac{1}{2}}\frac{\theta}{n}\right\} - 1}.$$

Secondly, suppose that $n = 1$; then the equation to the catenary will be, if $\theta = 0$ when $r = c$,

$$r = \frac{c}{1-\theta^2}.$$

Thirdly, let n be less than unity; then, if as before $\theta = 0$ when $r = c$, the equation will be

$$\epsilon^{(1-n^2)^{\frac{1}{2}}\frac{\theta}{n}} + \epsilon^{-(1-n^2)^{\frac{1}{2}}\frac{\theta}{n}} = \frac{2}{nr}\{r-(1-n)c\}.$$

Again, from (1, c) we have, since $C = c\tau$,

$$p = \frac{c\tau}{t},$$

and therefore by (a)

$$p = \frac{c\tau}{\tau + mkc - \frac{mkc^2}{r}} = \frac{nc}{n+1-\frac{c}{r}};$$

hence, putting $r = \infty$, we have $p = \frac{nc}{n+1}$, which shews that the three catenaries, corresponding to the three values of n, have all of them asymptotes passing within a distance $\frac{nc}{n+1}$ from the

centre of force. Put $r = \infty$ in the equations to the three curves, and we get for the inclinations of the pair of asymptotes of each to the line SO,

$$\frac{n}{(n^2-1)^{\frac{1}{2}}} \cos^{-1}\frac{1}{n}, \text{ and } \frac{n}{(1-n^2)^{\frac{1}{2}}} \log\frac{1+(1-n^2)^{\frac{1}{2}}}{n}.$$

John Bernoulli; *Opera*, Tom. IV. p. 240.
Whewell's *Mechanics*, 3rd edit. p. 183.

(3) To find the equation to a uniform catenary AOB (fig. 78), acted on by a central force tending to S, the intensity of which varies as the μ^{th} power of the distance; the tension at O being $(1+\mu)^{\text{th}}$ of the weight of a length SO of the string, each element of which length is supposed to be acted on by a constant force equal to that at O and towards S.

The notation remaining the same as in (2), the equation to the catenary will be

$$\left(\frac{c}{r}\right)^{\mu+2} = \cos(\mu+2)\,\theta.$$

(4) To find the equation to a uniform catenary $SAOB$, (fig. 79), acted on by a central repulsive force emanating from S, at which the two ends of the string are fastened, the intensity of this force varying inversely as the μ^{th} power of the distance; the tension at O being $(\mu-1)^{\text{th}}$ of the weight of a length SO of the string, each element of which length is supposed to be acted on by a constant force equal to that at O and from S.

The notation remaining the same as before, the equation to the curve will be

$$\left(\frac{r}{c}\right)^{\mu-2} = \cos(\mu-2)\,\theta.$$

SECT. 4. *Constrained Equilibrium.*

(1) A flexible string ab, (fig. 80), acted on by gravity, rests on the arc of a curve APB in a vertical plane; to find the tension of the string and the pressure on the curve at any point.

Let P, p, be any two points of the curve very near to each other; PO, pO, normals at these points, the point O being the centre of curvature when p approaches indefinitely near to P: let ax, ay, be the axes of x, y, the former being horizontal, the latter vertical; $aP = s$, $Pp = ds$; $t =$ the tension at P and $t + dt$ at p; $R =$ the unit of pressure on the curve at P, $m =$ the unit of mass of the string, $\angle POp = \phi$, $PO = \rho$.

Then, resolving forces, which act on the element Pp of the string, parallel to the tangent at P, we have

$$(t + dt)\cos\phi - t = mgds \, . \, \frac{dy}{ds},$$

or, neglecting infinitesimals of higher orders than the first,

$$dt = mgdy;$$

integrating and observing that t is equal to zero when $y = 0$, we get

$$t = mgy \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which gives the tension at any point of the string.

Again, resolving the forces on the element Pp parallel to the normal OP,

$$mgds \, . \, \frac{dx}{ds} + (t + dt)\sin\phi = Rds,$$

or, neglecting infinitesimals of orders higher than the first,

$$mg\frac{dx}{ds} + t\frac{\phi}{ds} = R;$$

but $\dfrac{\phi}{ds}$ is equal to $\dfrac{1}{\rho}$; hence we have for the pressure on the curve at any point,

$$R = mg\frac{dx}{ds} + \frac{t}{\rho}$$

$$= mg\left(\frac{dx}{ds} + \frac{y}{\rho}\right).$$

(2) Two equal weights Q, Q, are suspended at the extremities of a flexible string hanging over a smooth curve in a vertical plane; to find the pressure at any point of the curve, the weight of the string being reckoned inconsiderable.

Let APB (fig. 81) be the curve; OP, Op, normals at two consecutive points P, p; θ the inclination of OP to some assigned line in the plane of the curve, $POp = d\theta$; $PO = \rho$, $AP = s$, $Pp = ds$; $p =$ the unit of pressure on the curve at the point P; $t =$ the tension of the string at P, $t + dt =$ the tension at p.

Then for the equilibrium of the element Pp of the string we have, resolving forces at right angles to the tangent at P,

$$(t + dt) \sin d\theta = pds = p\rho d\theta,$$

and therefore, retaining infinitesimals of the first order,

$$t\, d\theta = p\rho d\theta, \quad t = p\rho \quad \text{.......................(1)}.$$

Again, resolving forces parallel to the tangent at P,

$$(t + dt) \cos d\theta - t = 0,$$

and therefore, retaining infinitesimals of the first order,

$$dt = 0, \; t = \text{constant};$$

but evidently at A the tension is equal to Q; hence $t = Q$. Hence from (1) we have

$$Q = p\rho, \; p = \frac{Q}{\rho}.$$

COR. The whole pressure on the curve AB is equal to

$$\int pds = Q\int \frac{ds}{\rho} = Q\int_{\theta_1}^{\theta_2} d\theta = Q\,(\theta_2 - \theta_1).$$

If the tangents at the points where the string leaves the curve be vertical, we have πQ for the whole pressure along the curve; if they be not vertical there will of course be pressures at the points A, B, in addition to the pressure along the curve.

Euler; *Nov. Comment. Petrop.* 1775, p. 307.
Poisson; *Traité de Mécanique*, Tom. I. ch. 3.

(3) To find the pressure on a curve AB, (fig. 82), when two weights Q, R, balance each other over it by means of a fine string, the friction between the string and the curve being taken into account; and the weight Q being considered as much greater than R as is consistent with equilibrium.

Let μ be the coefficient of friction; the rest of the notation being the same as in the preceding problem. Then the friction on the element Pp will be $\mu p ds$, and will act nearly in the direction of the tangent at P. Hence, resolving forces on the element Pp parallel to PO, we have

$$(t + dt) \sin d\theta = p ds = p\rho d\theta;$$

and therefore in the limit

$$t d\theta = p\rho d\theta, \quad t = p\rho \text{.....................} (1);$$

again, resolving forces parallel to the tangent at P,

$$(t + dt) \cos d\theta - t + \mu p ds = 0,$$

and therefore in the limit

$$dt + \mu p ds = 0,$$

and consequently by (1)

$$dt + \frac{\mu t}{\rho} ds = 0;$$

integrating, we get

$$\log t = -\mu \int \frac{ds}{\rho} = -\mu \int d\theta = C - \mu\theta \text{.........} (2);$$

hence, the values of t at A and B being Q and R,

$$\log Q = C - \mu\theta_1, \quad \log R = C - \mu\theta_2 \text{..........} (3),$$

and therefore $\log \dfrac{Q}{R} = \mu(\theta_2 - \theta_1), \quad \dfrac{Q}{R} = \epsilon^{\mu(\theta_2 - \theta_1)}$,

which expresses the relation which must subsist between Q and R under the circumstances of the problem.

Also, from (2) and (3),

$$\log \frac{t}{R} = \mu(\theta_2 - \theta), \quad t = R\epsilon^{\mu(\theta_2 - \theta)} \text{............} (4);$$

hence the whole pressure along the curve is equal to

$$\int p ds = \int \frac{ds}{\rho} t, \text{ from (1)},$$

$$= \int t d\theta = R \int \epsilon^{\mu(\theta_2 - \theta)} d\theta = C - \frac{R}{\mu} \epsilon^{\mu(\theta_2 - \theta)};$$

but, when $\theta = \theta_1$, it is clear that the pressure along the curve is zero; hence

$$0 = C - \frac{R}{\mu} \epsilon^{\mu(\theta_2 - \theta_1)},$$

and therefore the whole pressure from θ_1 to θ_2, is equal to

$$\frac{R}{\mu} \{\epsilon^{\mu(\theta_2 - \theta_1)} - 1\}.$$

In addition to this pressure along the curve there are the pressures at the extremities A and B.

Cor. If the curve be a semicircle $\theta_2 - \theta_1 = \pi$, and we have

$$\frac{Q}{R} = \epsilon^{\mu\pi}.$$

Euler; *Nov. Comment. Petrop.* 1775, p. 316.
Poisson; *Traité de Mécanique*, Tom. I. ch. 3.

(4) A heavy chain ABC (fig. 83), rests partially on the circumference of a circular section of a rough horizontal cylinder, moveable about its axis: having given the lengths of the two portions AB, BC, of the chain, to determine the moment of a force about the axis of the cylinder which shall maintain the equilibrium.

Let $AB = a$, $BC = b$, $r =$ the radius of the cylinder, and $m =$ the mass of a unit of the chain's length. Then the required moment is equal to

$$mrg\left(r \sin \frac{a}{r} + b\right).$$

(5) Two equal weights P, P', are connected by a string which passes over a rough fixed horizontal cylinder: to compare the forces required to raise P, accordingly as P is pushed up or P' pulled down.

If p be the force in the former and p' in the latter case,

$$\frac{p'}{p} = \epsilon^{\mu\pi}.$$

(6) If a weight P attached to one end of a fine cord, which is laid over a rough horizontal cylinder, can support a weight

nP attached to the other end, to determine the weight which it can support when the cord is wrapped r times round the cylinder.

The required weight is equal to

$$n^{2r+1}.P.$$

(7) A light thread, the length of which is $7a$, has its extremities fastened to those of a uniform heavy rod, the length of which is $5a$; and, when the thread is passed over a thin round peg, it is found that the rod will hang at rest provided that the point of support be anywhere within a space a in the middle of the thread: to determine the coefficient of friction between the thread and the peg; and, when the rod hangs in a position bordering upon motion, to find its inclination to the horizon and the tensions of the two parts of the string.

If μ represent the coefficient of friction, W the weight of the rod, S, T, the tensions of the longer and shorter parts of the string respectively, and θ the inclination of the rod to the horizon;

$$\mu = \frac{2}{\pi}\log\tfrac{4}{3}, \quad S = \tfrac{4}{5}W, \quad T = \tfrac{3}{5}W, \quad \cos\theta = \tfrac{24}{25}.$$

(8) A thin inextensible cord, in which the density of the material increases in geometric as the distance from one extremity increases in arithmetic progression, is laid directly across a rough horizontal cylinder, the circumference of a vertical section of which is equal to twice the length of the cord: to determine the coefficient of friction, supposing the cord to be only just supported when its two extremities are both in the horizontal plane through the axis of the cylinder.

Taking, in accordance with the hypothesis, $\alpha\epsilon^{\frac{s}{k}}$ as the expression for the density at a distance s from one end of the string, and denoting the radius of the cylinder by r,

$$\mu = \frac{r}{2k}.$$

SECT. 5. *Extensible Strings.*

If an extensible string of given length be stretched by any force, it is found by experiment that the extension of the string beyond its natural length is proportional to the force. From this it is easily seen that, if the string be of variable length, the extension will vary as the product of the force and the natural length of the string. Hence, if a denote the natural length of the string, and a' the length under the action of a stretching force P, we shall have

$$a' = a(1 + \lambda P),$$

where λ is a constant quantity depending upon the quality of the string, called the modulus of elasticity.

This theory was first announced by Hooke, in the form of an anagram, among a list of inventions at the end of his *Descriptions of Helioscopes*, published in the year 1676. The anagram is *ceiiinosssttuu*, from which may be extracted the proposition, "ut tensio sic vis." He afterwards published a work entitled *De Potentia Restitutiva or Spring*, in which the theory was developed at large with experimental illustrations. Hooke's theory forms the basis of a memoir by Leibnitz, in the *Acta Eruditorum* for the year 1684, entitled *Demonstrationes Novæ de Resistentia Solidorum*. For additional information on the subject the student is referred to s'Gravesande's *Element. Physic.* Lib. I. c. 26.

(1) An elastic string AC (fig. 84) is suspended from its extremity A, and has a weight attached to it at a point B; the natural lengths of AB, BC, being given, to find the length of the string AC in its present circumstances.

Let m denote the unit of mass of the string in its natural state; a, b, the natural lengths of AB, BC, and a', b', their lengths under the circumstances of the problem; c the length of a portion of the natural string, the weight of which is equal to the weight attached to B; let P be any point in AB, and p a point very near to it; let $AP = x$, $Pp = dx$; let t be the tension at P and $t + dt$ at p.

Then, since by Hooke's Principle the unit of mass of the element dx must evidently be diminished in the ratio of $1+\lambda t : 1$, the weight of Pp will be

$$\frac{mgdx}{1+\lambda t},$$

and therefore, for the equilibrium of Pp,

$$t + dt + \frac{mgdx}{1+\lambda t} - t = 0,$$

$$(1+\lambda t)\, dt + mgdx = 0,$$

integrating we get

$$t\,(1+\tfrac{1}{2}\lambda t) + mgx = C;$$

but it is evident that

$$t = mg\,(a+b+c), \quad \text{when } x = 0,$$

and

$$t = mg\,(b+c), \quad \text{when } x = a';$$

hence we obtain

$$(a+b+c)\,\{1+\tfrac{1}{2}\lambda mg\,(a+b+c)\} = (b+c)\,\{1+\tfrac{1}{2}\lambda mg\,(b+c)\} + a',$$

and therefore

$$\begin{aligned} a' &= a + \tfrac{1}{2}\lambda mg\,\{(a+b+c)^2 - (b+c)^2\} \\ &= a + \tfrac{1}{2}\lambda mg\,(a^2 + 2ab + 2ac) \\ &= a\,\{1 + \tfrac{1}{2}\lambda mg\,(a + 2b + 2c)\} \;\ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

Again, if Q be any point in BC, $BQ = y$, and τ = the tension at Q, we shall have, as before,

$$\tau\,(1+\tfrac{1}{2}\lambda t) + mgy = C;$$

but evidently

$$\tau = mgb, \quad \text{when } y = 0,$$

and

$$\tau = 0, \quad \text{when } y = b';$$

hence we have

$$b' = b\,(1+\tfrac{1}{2}\lambda mgb) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence from (1) and (2), if l' denote the whole length of the string AC, we find that

$$l' = a + b + \tfrac{1}{2}\lambda mg\,\{a\,(a+2b+2c) + b^2\}.$$

(2) **An elastic string, of which the unstretched length is a, is fixed at one end to the summit of a smooth inclined plane the length of which is also equal to a; to find the length which will hang over the plane, the string being stretched by its own weight.**

Let ACD (fig. 85) be the string hanging from the point A in the inclined plane AC; P any point in AC, and p a point near to P; let t be the tension at P, T at A, and τ at C; let $AP=x$, $Pp=dx$; $m=$ the unit of mass of the string when unstretched, $\alpha=$ the inclination of AC to the horizon.

Then, by virtue of Hooke's Principle, the mass of Pp will be

$$\frac{mdx}{1+\lambda t};$$

and therefore, $t+dt$ being the tension at p, we have for the equilibrium of Pp

$$dt+\frac{mg\sin\alpha}{1+\lambda t}dx=0,$$

$$(1+\lambda t)\,dt+mg\sin\alpha\,dx=0;$$

integrating we obtain

$$t\,(1+\tfrac{1}{2}\lambda t)+mgx\sin\alpha=C;$$

hence, τ being the value of t when $x=a$ and T when $x=0$, we have

$$\tau\,(1+\tfrac{1}{2}\lambda\tau)+mga\sin\alpha=T(1+\tfrac{1}{2}\lambda T),$$

$$(\tau-T)\,\{1+\tfrac{1}{2}\lambda\,(\tau+T)\}+mga\sin\alpha=0\ldots\ldots\ldots\ldots\ldots\ (1).$$

Let s be the natural length of CD; then $a-s$ will be the natural length of AC; hence clearly

$$\tau=mgs,\quad T=mg\,\{s+(a-s)\sin\alpha\};$$

we have then, by (1),

$$(a-s)\,[1+\tfrac{1}{2}\lambda mg\,\{2s+(a-s)\sin\alpha\}]=a,$$

$$\tfrac{1}{2}\lambda mg\,(a-s)\,\{2s+(a-s)\sin\alpha\}=s,$$

whence s may be determined by the solution of a quadratic equation.

If s' be the actual length of the portion CD of the string, we may shew that

$$s'=s\,(1+\tfrac{1}{2}\lambda ms),$$

and therefore s and s' are both known.

(3) A slightly extensible string Aa (fig. 86) is attached to the upper extremity A of the vertical radius AO of a circular arc AB along which it rests; having given its natural length, to find its length as it rests on the arc.

Let P, p, be any two points very near together in the string Aa; draw the lines PO, pO; let $AP = s$, $Pp = ds$, $\angle AOP = \phi$, $\angle POp = d\phi$, $AO = a$; $m =$ the unit of mass of the string when unstretched; let t, $t + dt$, be the tensions at P, p; s', ds', the lengths of AP, Pp, without stretching.

Then, by Hooke's Principle,

$$ds = (1 + \lambda t)\, ds' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, for the equilibrium of the portion Pp of the string, we have, resolving forces parallel to the tangent at P,

$$(t + dt) \cos (d\phi) + mgds' \sin \phi = t;$$

and therefore, retaining infinitesimals of the first order,

$$dt + mgds' \sin \phi = 0;$$

hence, by the aid of (1), we have

$$(1 + \lambda t)\, dt + mg \sin \phi\, ds = 0;$$

or, since $s = a\phi$,

$$(1 + \lambda t)\, dt + mag \sin \phi\, d\phi = 0;$$

integrating we get

$$t\,(1 + \tfrac{1}{2}\lambda t) - mag \cos \phi = C;$$

let β be the angle subtended at O by the arc Aa; then it is clear that, when $\phi = \beta$, t will be equal to zero; hence

$$-mag \cos \beta = C,$$

and therefore $\quad t\,(1 + \tfrac{1}{2}\lambda t) = mag\,(\cos \phi - \cos \beta) \ldots\ldots\ldots\ldots (2).$

From (1) we have, putting $a\phi$ for s,

$$ad\phi = (1 + \lambda t)\, ds',$$

and, λ being by the hypothesis a small quantity,

$$ds' = ad\phi\,(1 - \lambda t) = ad\phi - a\lambda t d\phi \ldots\ldots\ldots\ldots\ldots (3).$$

Now from (2) we get approximately

$$t = mag\,(\cos \phi - \cos \beta),$$

and therefore, substituting this value of t in the small term of the equation (3),

$$ds' = ad\phi - ma^2\lambda g\,(\cos\phi - \cos\beta)\,d\phi\,;$$

integrating we get

$$s' + C = a\phi - ma^2\lambda g\,(\sin\phi - \phi\cos\beta)\,;$$

but, when $\phi = 0$, it is evident that $s' = 0$; hence $C = 0$, and we have

$$s' = a\phi - ma^2\lambda g\,(\sin\phi - \phi\cos\beta)\,;$$

let $a\beta'$ be equal to the natural length of $A\alpha$; then evidently

$$a\beta' = a\beta - ma^2\lambda g\,(\sin\beta - \beta\cos\beta),$$

$$\beta' = \beta - ma\lambda g\,(\sin\beta - \beta\cos\beta)\,;$$

but, since $\beta = \beta'$ nearly, we may substitute β' for β in the coefficient of the small quantity λ; thus we obtain

$$\beta = \beta' + ma\lambda g\,(\sin\beta' - \beta'\cos\beta'),$$

which determines the required length $A\alpha$.

(4) Two weights P, Q, (fig. 87), resting on two smooth inclined planes CA, CB, are connected by a given elastic string PQ; to find their position of equilibrium.

Let θ be the inclination of QP to the horizon; α, β, the inclinations to the horizon of the planes CA, CB; a the natural length of the string PQ. Then the position of equilibrium will be defined by the two equations,

$$\tan\theta = \frac{P\cot\beta - Q\cot\alpha}{P+Q}, \quad PQ = a\left\{1 + \frac{\lambda P\sin\alpha}{\cos(\alpha-\theta)}\right\}.$$

(5) Two equal weights P, Q, (fig. 88), are connected by a fine elastic string PQ, of which the horizontal line BC is the natural length; to find the nature of the curves BP, CQ, on which they will always remain in equilibrium with the string parallel to the horizon, the plane of the curves being vertical.

Bisect BC in A, and draw AM vertical; let $AB = a = AC$, $AM = x$, $MP = y = MQ$; then the equation to each of the curves will be

$$(y-a)^2 = 2\lambda aPx,$$

or BP, CQ, are two semi-parabolas of which B, C, are the vertices.

(6) An elastic ring BC is placed round a vertical cone and descends by its own weight; to find the position of equilibrium.

Let O (fig. 89) be the centre of the ring in its position of equilibrium; let $\angle OAB = \alpha$, $2\pi a =$ the natural length of the ring, and $W =$ its weight; then

$$BO = a\left(1 + \frac{\lambda}{2\pi} W \cot \alpha\right),$$

which determines the required position.

(7) A fine elastic string is tied round two equal cylinders, the surfaces of which are in contact and axes parallel, the string not being stretched beyond its natural length; one of the cylinders is turned through two right angles, so that the axes are again parallel: to find the tension of the string, supposing that a weight of one pound would stretch it to twice its natural length.

The tension of the string $= \dfrac{\pi - 2}{\pi + 2}$ in pounds.

(8) A heavy uniform rod AB is attached by a hinge at A to an upright rod AC; the points B and C are connected together by a fine elastic string which, when unstretched, forms the hypotenuse of an isosceles right-angled triangle, A being the right angle: to find the position of equilibrium of AB.

If θ denote either of the angles at B or C, W the weight of AB, and λ the modulus of elasticity, the oblique position of equilibrium of AB is defined by the equation

$$\cos\theta = \frac{1}{\sqrt{(2)} - \lambda . W}.$$

There is therefore no oblique position unless

$$\lambda < \frac{\sqrt{(2)} - 1}{W}.$$

(9) A fine elastic string, passing over a smooth tack, supports a uniform rod, to the extremities of which the ends of the string are attached. Supposing the increase of the length of the string,

when stretched in a straight line by a force equal to the weight of the rod, to be equal to twice the length of the rod, to determine the position of equilibrium under the present circumstances.

If $2a$ = the natural length of the string, $2b$ = the length of the rod, and θ = the angle included between the two parts of the string,

$$\sin\theta = \frac{2b}{a^2}\{(a^2+b^2)^{\frac{1}{2}} - b\}.$$

(10) Two equal rigid rods AC, BC, without weight, are connected together by a smooth hinge at C and rest in a vertical plane, their lower extremities, which are tied together by an elastic string AB, being placed upon a smooth horizontal plane. If α be the inclination of each rod to the horizon, when a weight W is fixed to the middle point of each, and α' the inclination when a weight W' is so fixed; to find the natural length of the string.

If a be the length of each rod, the natural length of the string is equal to

$$2a \cdot \frac{W\sin\alpha' - W'\sin\alpha}{W\tan\alpha' - W'\tan\alpha}.$$

(11) Two fine strings, slightly elastic, are fastened to the middle points of the sides of a uniform rectangular board, thus crossing the board parallel to its sides, and intersecting in the centre. Supposing the board to be suspended from the intersection of the strings, to find the distance at which it will hang below the point of suspension.

If W be the weight of the board, $2a$, $2b$, the lengths of its sides, and λ the modulus of the elasticity of the strings, the required distance is equal to

$$\left(\frac{\mu W}{\frac{1}{a^3}+\frac{1}{b^3}}\right)^{\frac{1}{3}}.$$

(12) Six equal rods are connected together by hinges at their ends so as to form a hexagon, and, one of the rods being supported in a horizontal position, the opposite one is fastened to it

by a fine elastic string joining their middle points. Supposing the modulus of elasticity to be equal to the reciprocal of the weight of each rod, to find the original length of the string in order that the hexagon may be equiangular in the position of equilibrium.

If $a =$ the length of each rod, and $l =$ the natural length of the string,

$$l = \frac{\sqrt{3}}{4} \cdot a.$$

(13) A heavy elastic string is laid upon a smooth double inclined plane in such a manner as to remain at rest: to find how much the string is stretched.

If $W =$ the weight of the string, $c =$ its natural length, and α, α', denote the inclinations of the planes; then the required extension is equal to

$$\tfrac{1}{2}\lambda Wc \cdot \frac{\sin\alpha \sin\alpha'}{\sin\alpha + \sin\alpha'}.$$

CHAPTER VI.

VIRTUAL VELOCITIES.

THE Principle of Virtual Velocities consists in the following general proposition:

"If any assignable system of bodies or points, solicited each of them by any forces whatever, be in equilibrium; and we conceive this system to experience consistently with its geometrical relations any small arbitrary displacement, by virtue of which each point describes an indefinitely small space; the sum of the forces multiplied each of them by the resolved part, parallel to its direction, of the space described by its point of application, will be always equal to zero; this resolved part being considered positive when it lies in the direction of its corresponding force, and negative when in an opposite direction."

The resolved parts of the spaces described by the points of application of the forces are called their Virtual Velocities. Let P, Q, R,......denote any system of forces acting on a system of points consistently with equilibrium; and let α, β, γ,...... denote their virtual velocities; then, as far as the first powers of α, β, γ,......are concerned,

$$P\alpha + Q\beta + R\gamma + S\delta + \ldots\ldots = 0 \ldots\ldots\ldots \text{(A)}.$$

The Principle of Virtual Velocities was first detected by Guido Ubaldi[1] as a property of the equilibrium of the lever and of moveable pulleys. Its existence was afterwards recognized by Galileo[2] in the inclined plane, and the machines depending upon it. The expression 'moment' of a force or weight acting on any machine, was used by Galileo to denote its energy or effort to set the machine in motion, who accordingly declared that for the

[1] *Mechanicorum Liber; De Libra, De Cochlea.*

[2] *Della Scienza Mecanica, Opera*, Tom. I. p. 265; *Bologna*, 1655.

equilibrium of a machine acted on by two forces, it is necessary that their moments should be equal, and should take place in opposite directions; he shewed moreover that the moment of a force is always proportional to the force multiplied by its virtual velocity. The word 'moment' was used in the same sense by Wallis[1], who adopted Galileo's principle of the equality of moments as the fundamental principle of Statics; and deduced from it the conditions for the equilibrium of the principal machines. Descartes[2] has likewise reduced the whole science of Statics to a single principle, which virtually coincides with that of Galileo; it is presented however under a less general aspect. The principle is, that it requires precisely the same force to raise a weight P through an altitude a, as a weight Q through an altitude b, provided that P is to Q as b to a. From this it follows, that two weights attached to a machine will be in equilibrium when they are disposed in such a manner that the small vertical paths which they can simultaneously describe are reciprocally as the weights.

Torricelli[3] is the author of another principle which may be immediately deduced from the principle of virtual velocities: the principle is, that when any two weights rigidly connected together are so placed that their centre of gravity is in the lowest position which it can assume consistently with the geometrical conditions to which they are subject, they will be in equilibrium. The principle of Torricelli has given birth to the following more general one, viz.—that any system whatever of heavy bodies will be in equilibrium when their centre of gravity is in its lowest or highest position.

John Bernoulli was the first to announce the principle of virtual velocities under its most general aspect in the form which we have given above, in a letter to Varignon[4], dated Bâle, Jan. 26, 1717. The striking value of the principle, as an instrument of analytical generalization, has been splendidly exhibited by Lagrange in his *Mécanique Analytique*.

[1] *Mechanica, sive de Motu, Tractatus Geometricus.*

[2] *Lettre* 73, Tom. I. 1657; *de Mechanica Tractatus, Opuscula Posthuma.*

[3] *De Motu gravium naturaliter descendentium*, 1644.

[4] *Nouvelle Mécanique*, Tom. II. sect. 9.

From the principle of virtual velocities may be immediately deduced the principle which was proposed by Maupertuis in the *Mémoires de l'Académie des Sciences de Paris* for the year 1740, under the name of the *Loi de Repos;* and which Euler has developed at large in the *Mémoires de l'Académie de Berlin* for the year 1751. Suppose that any number of forces P, Q, R,...tending towards fixed centres and functional of their distances p, q, r, ...from the centres, to act on a system of points rigidly connected together. Then supposing the system of points to be slightly displaced, so that p, q, r,...receive increments dp, dq, dr,...we shall have, by the principle of virtual velocities,

$$Pdp + Qdq + Rdr + \ldots = 0.$$

Let $d\Pi$ denote the left-hand member of this equation; then

$$d\Pi = 0 \ldots\ldots\ldots \text{(B)}.$$

From this it appears that if the system be so placed that Π may have a maximum or a minimum value, there will be equilibrium: this proposition constitutes Maupertuis' Principle of Rest. It does not however follow conversely that, whenever the system is at rest, Π shall have a maximum or minimum value, since by the principles of the differential calculus we know that the equation (B), although a necessary, is not the only condition for the existence of such a value. Lagrange[1] has shewn that if Π be a minimum the equilibrium will be stable, and if a maximum, unstable.

As an example of this theory, it is evident that, if any system be in equilibrium under the action of gravity, there will be stable or unstable equilibrium accordingly as the centre of gravity is in the lowest or highest position which is compatible with the geometrical relations to which the system is subject.

The principle of equilibrium developed by Courtivron[2] is likewise grounded upon the principle of virtual velocities; Courtivron's Principle asserts, that if a system of bodies be in motion under the action of any forces varying according to any assigned laws, a position of the system corresponding to a maxi-

[1] *Mécanique Analytique. Première Partie*, sect. 5.

[2] *Mémoires de l'Académie des Sciences de Berlin*, 1748, 1749.

mum or minimum value of the *vis viva* will be a position of equilibrium; a maximum value of the *vis viva* corresponding to stable, and a minimum to unstable equilibrium.

Sect. 1. *Equilibrium.*

(1) A particle P (fig. 90) is attracted towards two centres of force A and B; to find the position of the particle that it may be in equilibrium.

Let A, B, denote the two forces; let $AP = r$, $BP = s$, $AB = a$; draw PM at right angles to AB, and let $AM = x$, $PM = y$. Then, supposing P to receive some slight arbitrary displacement, the decrements dr, ds, of r, s, will be the virtual velocities of the forces A, B; hence, by the formula (A),

$$Adr + Bds = 0 \quad \text{.......................} \quad (1).$$

But
$$r = (x^2 + y^2)^{\frac{1}{2}}, \quad s = \{(a-x)^2 + y^2\}^{\frac{1}{2}},$$

$$dr = \frac{xdx + ydy}{(x^2+y^2)^{\frac{1}{2}}}, \quad ds = \frac{-(a-x)\,dx + ydy}{\{(a-x)^2 + y^2\}^{\frac{1}{2}}};$$

and therefore, by (1),

$$A\frac{xdx + ydy}{(x^2+y^2)^{\frac{1}{2}}} + B\frac{-(a-x)\,dx + ydy}{\{(a-x)^2 + y^2\}^{\frac{1}{2}}} = 0;$$

but since dx and dy are independent quantities, whatever be the small variation in the position of P, we have, equating their coefficients to zero,

$$\frac{Ax}{(x^2+y^2)^{\frac{1}{2}}} - \frac{B(a-x)}{\{(a-x)^2 + y^2\}^{\frac{1}{2}}} = 0 \quad \text{................} \quad (2),$$

$$\frac{Ay}{(x^2+y^2)^{\frac{1}{2}}} + \frac{By}{\{(a-x)^2 + y^2\}^{\frac{1}{2}}} = 0 \quad \text{................} \quad (3).$$

From (3) we have $y = 0$, and therefore from (2) we see that $A = B$; thus it appears that if any particle be acted on by two forces tending towards two fixed centres, the conditions for its

equilibrium are, first, that it shall lie in the straight line joining the two centres, and, secondly, that the two forces shall be equal.

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 184.

(2) A rigid rod AB (fig. 91) without weight, rests over a peg O, and against a smooth wall CD, and is acted on by a weight P suspended from the extremity A; to determine its position of equilibrium and the pressures on the wall and the peg.

Draw EOF horizontally; let $AB=a$, $OB=x$, $OE=b$, $AF=y$. Let R, S, denote the reactions of the wall and peg against the rod, of which the former will be horizontal, and the latter at right angles to AB. Conceive the rod AB to be slightly displaced from its position of rest by making its end B slide along CD, the peg O still touching the rod; then it is evident that the point B will have no motion parallel to R, and that the motion of the point O of the rod resolved parallel to S will be an infinitesimal of the second order. Hence of the three forces P, S, R, P alone will have a virtual velocity. We have then, by the principle of virtual velocities,

$$Pdy=0, \text{ or } dy=0 \ldots\ldots. (1).$$

Now, by similar triangles AFO, BEO, there is

$$AF=AO\,.\frac{BE}{BO},$$

and therefore

$$y=\frac{a-x}{x}(x^2-b^2)^{\frac{1}{2}};$$

differentiating this equation and performing obvious simplifications, we shall have

$$dy=\frac{ab^2-x^3}{x^2(x^2-b^2)^{\frac{1}{2}}}dx;$$

and therefore, by (1),

$$ab^2-x^3=0, \quad x=(ab^2)^{\frac{1}{3}} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which defines the position of equilibrium.

In order to determine S, conceive each point of the rod to receive the same vertical displacement β, the point B thus sliding along CD and the rod moving parallel to itself.

Then, putting $\angle BOE = \phi$, the virtual velocities of P, R, S, will be $-\beta, 0, \beta \cos \phi$, respectively, and therefore

$$S \, . \, \beta \cos \phi = P \, . \, \beta, \quad S \cos \phi = P;$$

but $$\cos \phi = \frac{b}{x} = \left(\frac{b}{a}\right)^{\frac{1}{3}}, \text{ by (2)};$$

hence $$S = P\left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

In order to find R, conceive the rod to be displaced along its length through a space β; then, the virtual velocities of P, R, S, being $-\beta \sin \phi$, $\beta \cos \phi$, 0, respectively, we have

$$R\beta \cos \phi = P\beta \sin \phi, \quad R = P \tan \phi;$$

but $$\tan \phi = \frac{(x^2 - b^2)^{\frac{1}{2}}}{b} = \frac{(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}, \text{ by (1)};$$

therefore $$R = P\frac{(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 196.

(3) A particle is placed upon a smooth inclined plane AB, (fig. 92), at a point O, and acted on by a force P in a given direction; to determine the magnitude of P, that the particle may be at rest, and the pressure on the plane.

Let W be the weight of the particle, R the reaction of the plane, $\angle POB = \epsilon$, $\alpha =$ the inclination of AB to the horizon AC.

Conceive the particle O to receive a displacement β along the plane AB; then, the virtual velocity of R being zero, the virtual velocities of P, W, will be $\beta \cos \epsilon$, $-\beta \sin \alpha$, respectively. Hence, by the principle of virtual velocities,

$$P\beta \cos \epsilon = W\beta \sin \alpha, \; P \cos \epsilon = W \sin \alpha \; \ldots\ldots \; (1),$$

which determines the value of P.

Next, displace the particle parallel to AC through a space β; then, the virtual velocity of W being zero, the virtual velocities of P, R, will be respectively $\beta \cos (\alpha + \epsilon)$, $-\beta \sin \alpha$, and therefore

$$P\beta \cos (\alpha + \epsilon) = R\beta \sin \alpha,$$

whence $$R=\frac{P\cos(\alpha+\epsilon)}{\sin\alpha}=\frac{W\cos(\alpha+\epsilon)}{\cos\epsilon}, \text{ by (1).}$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 191.

(4) A rigid rod OA, (fig. 93), without weight, is acted on by a weight P hanging from its extremity A; the end O of the rod is fixed; also EF is a spring in the form of a circular arc to a centre O, of which the force of contraction varies as the angle AOB, OB being a horizontal line; to find the position of the rod that it may be at rest.

Let $OA=a$, $\angle AOB=\phi$, $OF=b$; let α be the value of ϕ when the force of the spring's contraction is equal to E; then, corresponding to the angle ϕ, the force of contraction will be equal to

$$\frac{E\phi}{\alpha}.$$

Let AO be displaced slightly through an angle $d\phi$ into the position Oa; draw ap at right angles to AP, and let f be the new position of F; then, by the principle of virtual velocities,

$$P.Ap-\frac{E}{\alpha}\phi.Ff=0\text{.......................(1)};$$

but, since Ap, Aa, are respectively at right angles to OB, OA, it is clear that $\angle aAp=\phi$, and therefore

$$Ap=Aa\cos\phi=ad\phi.\cos\phi;$$

also $$Ff=bd\phi:$$

hence from (1) we have

$$Pa\cos\phi-\frac{E}{\alpha}\phi b=0;$$

or $$\frac{\cos\phi}{\phi}=\frac{Eb}{Pa\alpha},$$

the required condition of equilibrium, from which ϕ is to be determined.

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 196.

(5) A smooth rod AB (fig. 94) rests against two horizontal bars which pierce the vertical plane through the rod at right angles in the points A', A''; the rod passes under the lower and over the higher bar, its lower extremity A being sustained upon

a smooth horizontal plane; to determine the pressures upon the two bars, and upon the horizontal plane.

The pressures upon the bars and upon the horizontal plane will be equal to their reactions upon the rod; the reactions of the bars upon the rod will be two forces R', R'', at right angles to the rod; and the reaction of the horizontal plane will be a vertical force R. Let G be the centre of gravity of the rod, at which point we will suppose its whole weight to be collected. Thus we have four forces R, R', R'', W, acting respectively at the four points A, A', A'', G, rigidly connected together, so as to produce equilibrium.

Let $AG = a$, $A'A'' = b$, and $\alpha =$ the inclination of the rod to the horizon.

Conceive the rod to receive a small displacement of such a character that it still remains in contact with the two bars; then evidently the virtual velocities of W and R will be equal, the one being a positive and the other a negative velocity, and therefore, a denoting the magnitude of the virtual velocity of each, we have

$$Ra - Wa = 0,$$

and therefore $$R = W \qquad (1).$$

Next, conceive the rod to receive a slight displacement, as in (fig. 95), by revolving through a small angle ω about the point A'' which is supposed to be kept stationary; the points a, a', g, being the new positions of A, A', G; from a draw am at right angles to the vertical line through A, and from g draw gn at right angles to the vertical Gn through G. Then, by the principle of virtual velocities,

$$R \,.\, Am - R' \,.\, A'a' - W \,.\, Gn = 0,$$

and therefore, by (1),

$$W(Am - Gn) - R' \,.\, A'a' = 0 \qquad (2);$$

but $$Am = Aa \cos\alpha = AA'' \,.\, \omega \,.\, \cos\alpha,$$

and $$Gn = Gg \cos\alpha = A''G \,.\, \omega \,.\, \cos\alpha;$$

and therefore $$Am - Gn = AG \,.\, \omega \cos\alpha = a\omega \cos\alpha;$$

also $$A'a' = A'A'' \,.\, \omega = b\omega.$$

Hence from (2) we have, substituting for $Am - Gn$ and $A'a'$ their values, $$Wa\omega \cos\alpha - b\omega R' = 0,$$

and therefore $$R' = \frac{Wa \cos \alpha}{b} \quad \text{......................} (3).$$

Again, conceive the rod, as in fig. 96, to be slightly displaced into the position $aa'a''b$, parallel to its original position, and still touching the horizontal plane; a, a', a'', b, being the new positions of A, A', A'', B. Then, the virtual velocities of R' and R'' being equal and opposite, we have

$$R'' = R' = W\frac{a \cos \alpha}{b}, \text{ by (3).}$$

(6) A string of given length passes over a given pulley; it has attached to its two extremities two weights, one of which is capable of sliding freely on a given curve; to determine the curve on which the other ought to slide in order that in every position of the two weights they may be in equilibrium.

Let P, P', (fig. 97), denote the two weights in any position; A the pulley; and let AB be a vertical line through A; let $AP = \rho$, $AP' = \rho'$. Draw PM, $P'M'$, at right angles to AB; let $AM = x$, $AM' = x'$, $\angle PAB = \phi$, $\angle P'AB = \phi'$.

Then, supposing the two weights to receive displacements along two small arcs of their corresponding curves of constraint, we have, by the principle of virtual velocities,

$$Pdx + P'dx' = 0;$$

and, since this relation is true for all corresponding points of the two curves, we have, integrating,

$$Px + P'x' = c,$$

where c is some constant quantity; and therefore, in polar coordinates,

$$P\rho \cos \phi + P'\rho' \cos \phi' = c \quad \text{.................} (1).$$

Again, if l denote the length of the string,

$$\rho + \rho' = l \quad \text{...........................} (2).$$

Supposing the curve on which P' moves to be the given one, we have $$f(\rho', \phi') = 0 \quad \text{..........................} (3),$$

where $f(\rho', \phi')$ denotes some known function of ρ', ϕ'.

Eliminate from the equations (1), (2), (3), the quantities ρ', ϕ', and we shall get for the equation of the required curve

$$\chi(\rho, \phi) = 0,$$

$\chi(\rho, \phi)$ denoting some function of ρ, ϕ.

John Bernoulli; *Act. Erudit.* 1695. Febr. p. 59. Leibnitz; *Ib.* April, p. 184. L'Hôpital; *Act. Erudit. Suppl.* Tom. II. sect. 6. p. 289. Fuss; *Nova Acta Acad. Petrop.* 1788, p. 197.

(7) Four uniform beams AB, BC, CD, DE, (fig. 98), connected together by smooth hinges, are placed in a position of equilibrium, the ends A and E being attached to two smooth hinges in the same horizontal line AE; the beam AB is equal to the beam ED, and the beam BC to the beam CD; to compare the angles BAE and CBD.

Let $AB = DE = 2a$, $BC = CD = 2b$, $\angle BAE = \alpha$, $\angle CBD = \beta$, $AE = c$; $h =$ the height of the centre of gravity of the beams above the line AE; $m =$ the weight of each of the lower and $n =$ that of each of the higher beams. Then

$$(2m + 2n)\, h = 2ma \sin\alpha + 2n\,(2a \sin\alpha + b \sin\beta),$$

$$(m + n)\, h = (m + 2n)\, a \sin\alpha + nb \sin\beta;$$

but for equilibrium h must be a maximum or a minimum; hence

$$0 = (m + 2n)\, a \cos\alpha\, d\alpha + nb \cos\beta d\beta \;\ldots\ldots\ldots\ldots\ldots (1).$$

Again, it is evident by the geometry that

$$c = 4a \cos\alpha + 4b \cos\beta,$$

and therefore $$0 = a \sin\alpha\, d\alpha + b \sin\beta\, d\beta \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiply (1) by $\sin\alpha \sin\beta$, and then, by (2), we have

$$(m + 2n) \cos\alpha \sin\beta \,.\, b \sin\beta d\beta = n \cos\beta \sin\alpha \,.\, b \sin\beta d\beta,$$

and therefore $$(m + 2n) \cos\alpha \sin\beta = n \cos\beta \sin\alpha,$$

$$\tan\alpha = \frac{m + 2n}{n} \tan\beta.$$

If $m = n$, we have

$$\tan\alpha = 3 \tan\beta.$$

(8) A beam AB, (fig. 99), rests with one end against a smooth vertical plane OK, and upon a smooth curve $\alpha\beta$; the plane of the beam and the curve being at right angles to the plane OK; to determine the nature of the curve, that the beam may rest in any position.

From any point O in the section OK of the vertical plane draw OL horizontal; from the point of contact P corresponding to any position of the beam draw PM vertical; let G be the centre of gravity of the beam; draw GH vertical. Let $OM = x$, $PM = y$, $AG = a$. Then, from the geometry, we see that

$$GH = y + PG.\frac{dy}{ds} = y + \left(a - x\frac{ds}{dx}\right)\frac{dy}{ds} = y - x\frac{dy}{dx} + a\frac{dy}{ds}.$$

Now since for the equilibrium of a material system acted on by gravity, it is necessary that its centre of gravity be in the highest or lowest possible position consistent with geometrical relations, it is clear that in the present problem, equilibrium being possible for every position of the beam, GH must be of invariable magnitude. Hence, if $p = \dfrac{dy}{dx}$,

$$C = y - xp + \frac{ap}{(1+p^2)^{\frac{1}{2}}};$$

differentiating with respect to x, and putting $\dfrac{dp}{dx} = q$,

$$0 = -xq + \frac{aq}{(1+p^2)^{\frac{3}{2}}};$$

hence $$q = 0, \quad \text{or } x = \frac{a}{(1+p^2)^{\frac{3}{2}}}:$$

the former of these solutions gives a straight line for the locus of P; if we integrate the second equation, we shall get, for the equation to the locus of P,

$$C = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} + y.$$

Suppose the origin of co-ordinates to be so chosen that $y = 0$ when $x = a$, in which case O will be the intersection of the beam

in its horizontal position with the line OK; then $C=0$, and the equation will be

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

(9) A particle O is acted upon by three forces A, B, C, passing through three points A, B, C; to determine the conditions for the equilibrium of the particle by the principle of virtual velocities.

The three points A, B, C, must all lie in a single plane containing the particle; also the relative magnitude of the forces A, B, C, are given by any two of the three proportions,

$$B : C :: \sin COA : \sin AOB,$$

$$C : A :: \sin AOB : \sin BOC,$$

$$A : B :: \sin BOC : \sin COA.$$

Euler; *Mémoires de l'Académie de Berlin*, 1751, p. 185.

(10) A particle is acted on by any number of forces; to find the conditions to which their magnitudes and directions must be subject that the particle may be at rest.

From the particle draw straight lines representing the forces in magnitude and in direction; then, that the particle may be in equilibrium, its position must coincide with the centre of gravity of a number of equal particles placed at the extremities of the straight lines.

This celebrated theorem for the equilibrium of a particle is due to Leibnitz[1]: Euler[2] gave a demonstration by the aid of Maupertuis' *Loi de Repos*, and Lagrange[3] by the principle of Virtual Velocities. See also Poisson, *Traité de Mécanique*, Tom. I. No. 67. A more general theorem of forces, which comprehends this of Leibnitz as a particular case, has been given by Chasles[4]: see *Bulletins de l'Académie des Sciences et Belles-Lettres de Bruxelles*, 1840, 2me partie, p. 261.

[1] *Journal des Savans*, 1693; *Opera*, Tom. III. p. 283.
[2] *Mémoires de l'Académie de Berlin*, 1751.
[3] *Mécanique Analytique*, Tom. I. p. 106.
[4] *Correspondance Mathématique*, Tom. V. p. 106—108; 1829.

(11) A string of given length passes over a fixed point; it has attached to its two extremities two weights, one of which is capable of sliding freely along an inclined plane passing through the point; to determine the curve on which the other must be placed that in every position of the two weights they may be in equilibrium.

Let the angle which the inclined plane makes with the vertical be α; then, the notation remaining the same as in (6), the equation to the required curve will be

$$(P \cos \phi - P' \cos \alpha)\ \rho = c - P'l \cos \alpha,$$

which belongs to a conic section.

Fuss; *Nova Acta Acad. Petrop.* 1788.

(12) A beam PQ, (fig. 100), rests against a smooth vertical plane AB and a smooth curve AP; to find the nature of the curve that the beam may be at rest in all positions.

Let G be the centre of gravity of the beam; draw PM horizontal; let $PQ = a$, $GP = c$, $AM = x$, $PM = y$; then the equation to the curve will be

$$\frac{(c-x)^2}{c^2} + \frac{y^2}{a^2} = 1,$$

which is the equation to an ellipse, the centre of which coincides with Q when PQ is horizontal.

(13) A uniform beam AB, (fig. 101), rests upon a smooth horizontal plane Ca, and against a smooth vertical plane Cb; a string ACP is attached to the end A of the beam, and hangs through a small ring at C, with a weight P at its extremity; to find the position of the beam when at rest.

If W denote the weight of the beam, and θ the angle BAC, then

$$\tan \theta = \frac{W}{2P}.$$

(14) A plane figure, bounded by a parabola, rests in a vertical plane, on two points in the same horizontal line, the centre of gravity of the figure being in the axis of the parabola at a given distance from the vertex; to find the position of equilibrium.

Let $2a$ be the distance between the two points, $4m$ the latus rectum, h the distance of the centre of gravity from the vertex, and θ the inclination of the axis to the vertical in the position of equilibrium; then the equation

$$\sin\theta\,\{3a^2\cos^4\theta - 4m\,(h-m)\cos^2\theta + 4m^2\} = 0$$

will give the positions of equilibrium.

17 (15) A particle is attracted towards each of two fixed centres of force varying inversely as the square of the distance; to find the equation to the surface on which it may remain at rest in every position.

If μ, μ', be the absolute forces of attraction; r, r', simultaneous distances of the particle from the centres; and a, a', given values of r, r'; then the equation to the surface will be

$$\frac{\mu}{r} + \frac{\mu'}{r'} = \frac{\mu}{a} + \frac{\mu'}{a'}.$$

18 (16) To the extremity B of a rod AB, (fig. 102), which is able to revolve freely about A, is attached an indefinitely fine thread BCM, passing over a point C vertically above A, and sustaining a heavy particle at M on a smooth curve CMN in the vertical plane BAC; to determine the nature of the curve that for all positions of the rod and particle the system may be in equilibrium.

Let $AB = 2a$, $AC = b$, $l =$ the length of the thread BCM, $\rho =$ the straight line CM, $\theta = \angle ACM$, $m =$ the mass of the particle, $m' =$ the mass of the rod. Then the equation to the curve will be

$$m'\rho^2 + 2\,(2mb\cos\theta - m'l)\,\rho = c,$$

where c is a constant quantity.

This problem was proposed by Sauveur to L'Hôpital, by whom a solution was published in the *Acta Eruditorum*, 1695, Febr. p. 56. The curve was shewn by John Bernoulli, *Ib.* p. 59, to be an Epitrochoid. See also James Bernoulli, *Ib.* p. 65.

(17) A rod passing through a fixed ring in the vertical axis of a surface of revolution, rests in all positions with one end on the surface: to find the nature of the generating curve.

Let P be any point of the curve, O the ring, θ the inclination of OP to the vertical line drawn downwards from O; then, if $OP=r$, the equation to the curve is

$$r = a + c \sec\theta,$$

c being an arbitrary constant.

(18) The plane of a parabola is vertical and its axis horizontal: two weights are placed on the curve, being attached to the ends of a fine string which passes over a pulley at the focus: to find the condition of equilibrium.

If P, P', be the weights and y, y', their distances below the axis, the condition of equilibrium is expressed by the equation

$$\frac{P}{P'} = \frac{y}{y'}.$$

(19) A uniform rod of length l rests between the concave arc of an ellipse and the axis minor, which is vertical, the axes of the ellipse being $2l$ and l: to determine the position of the rod's equilibrium.

The rod will be in equilibrium at all inclinations to the horizon.

(20) Two weights, P, P', are attached to the ends of a string which hangs in contact upon a parabola of which the axis is vertical; to find the condition of equilibrium.

If x, x', represent the distances of the weights P, P', below the vertex, and $4m$ denote the latus rectum, the weights will rest if

$$\frac{P^2}{x'} - \frac{P'^2}{x} = \frac{P'^2 - P^2}{m}.$$

(21) To find a point on the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a particle, attracted towards the origin by any force, will remain at rest.

The point required is given by the equations

$$\frac{x}{a^3} = \frac{y}{b^3} = \frac{z}{c^3} = \frac{1}{(a^6 + b^6 + c^6)^{\frac{1}{2}}}.$$

(22) A right cylinder on an elliptic base, the semiaxes of which are a and b, rests with its axis horizontal between two smooth planes inclined at right angles to each other, the line of intersection of the planes being parallel to the axis of the cylinder: to determine its positions of equilibrium, (1) when the inclination of one of the planes is greater than $\tan^{-1}\dfrac{a}{b}$, (2) when the inclination of both planes is less than $\tan^{-1}\dfrac{a}{b}$.

Let α be the inclination of one of the planes to the horizon, and ϕ the inclination of the major axis of a transverse section of the cylinder to the other plane. Then, under the hypothesis (1), there will be two positions of equilibrium, viz. when the major-axis of the section is parallel to either of the two planes. Under the hypothesis (2), there will be three positions of equilibrium, viz. two the same as under the former hypothesis, and one as defined by the equation

$$\cos 2\phi = -\frac{a^2 + b^2}{a^2 - b^2} \cdot \cos 2\alpha.$$

SECT. 2. *Stability and Instability of Equilibrium.*

(1) Three weights are suspended from the angles of an isosceles triangle, the plane of which is vertical, and which is supported by a horizontal axis passing through its centre of gravity, about which it is able to revolve: to determine its positions of equilibrium, when the two weights suspended from the extremities of the base of the triangle are equal to each other, and are each of them greater than the third; and to determine the character of the equilibrium.

Let A (fig. 103) be the vertex, and AB the axis of the triangle, G being the centre of gravity. Let $AB = 3a$. Let P be the smaller weight, and Q either of the larger ones. The two

weights Q may be collected at B. Let θ be the inclination of AB to the vertical. The moment of $2Q$ about G is $2Q\,a \sin\theta$, and that of P, in an opposite direction, $P.2a\sin\theta$.

The resultant of these two moments is

$$2a\,(Q-P)\sin\theta,$$

estimated in the direction of the arrows. This moment, from $\theta=0$ to $\theta=\pi$, always acts in the same direction, provided that θ be not actually 0 or π, in which cases the moment vanishes. Hence we see that, for equilibrium,

$$\theta=0 \text{ or } \theta=\pi,$$

the former corresponding to stable and the latter to unstable equilibrium.

(2) AB (fig. 104) is a beam moveable about a hinge A; C is a small pulley in the vertical line through A, AC being equal to AG, where G is the centre of gravity of AB; a fine string is attached to G, which passes over C and has a weight P suspended by it; to find the stable and unstable positions of equilibrium of the beam.

Let $GA=CA=a$, $l=$ the length of the string GCP, $W=$ the weight of the beam AB, $\angle\,GCA=\theta$; $x=$ the vertical distance of the centre of gravity of P and the beam below the horizontal line through C.

Now, from the geometry,

$$CP=l-2a\cos\theta;$$

and the distance of G below the horizontal line through C, is

$$a+a\cos 2\theta;$$

hence, by the property of the centre of gravity of bodies,

$$(P+W)\,x=P\,(l-2a\cos\theta)+Wa\,(1+\cos 2\theta).$$

Now for equilibrium x must have a maximum or a minimum value; hence evidently

$$u=W\cos 2\theta-2P\cos\theta$$

must have a maximum or minimum value; therefore

$$\frac{du}{d\theta} = -2W \sin 2\theta + 2P \sin \theta = 0,$$

and therefore $\quad \sin \theta\,(2W \cos \theta - P) = 0;$

hence for equilibrium it is necessary that $\sin \theta = 0$, and therefore $\theta = 0$, or $\cos \theta = \dfrac{P}{2W}$.

Differentiating u a second time, we get

$$\frac{d^2u}{d\theta^2} = -4W \cos 2\theta + 2P \cos \theta;$$

if $\theta = 0$, we have $\quad \dfrac{d^2u}{d\theta^2} = -4W + 2P;$

hence $\dfrac{d^2u}{d\theta^2}$ will be positive or negative, and therefore u a minimum or a maximum according as P is greater or less than $2W$; hence, if P be greater than $2W$, $\theta = 0$ gives a position of unstable equilibrium, and, if P be less than $2W$, one of stability.

Again, if $\cos \theta = \dfrac{P}{2W}$, we shall have

$$\frac{d^2u}{d\theta^2} = -4W(2\cos^2\theta - 1) + 2P \cos \theta = \frac{4W^2 - P^2}{W};$$

if then $2W$ be greater than P, $\dfrac{d^2u}{d\theta^2}$ is positive, and therefore the altitude of the centre of gravity of P and the beam is a maximum, and therefore the position will be one of unstable equilibrium; if $2W$ be less than P, $\cos \theta$ will be impossible, or the only position of equilibrium will be the unstable one given by $\theta = 0$.

(3) A uniform beam PQ (fig. 105) is placed upon two smooth inclined planes AB, AC; to find whether its position of equilibrium is one of stability or of instability.

Let G be the centre of gravity of the beam; from P and G draw PM, GH, at right angles to the horizontal plane bAc through A. Let $\angle BAb = \alpha$, $\angle CAc = \beta$, $PG = QG = a$, $\theta =$ the

angle of inclination of PQ to the horizon, $GH = z$. Then, by the geometry,

$$z = a \sin \theta + PM = a \sin \theta + AP \sin \alpha$$

$$= a \sin \theta + \sin \alpha \,.\, 2a \frac{\sin (\beta - \theta)}{\sin (\alpha + \beta)}$$

$$= \frac{a}{\sin (\alpha + \beta)} \{\sin (\beta - \alpha) \sin \theta + 2 \sin \alpha \sin \beta \cos \theta\};$$

then, if z be a maximum or minimum,

$$u = \sin (\beta - \alpha) \sin \theta + 2 \sin \beta \sin \alpha \cos \theta$$

will be a maximum or minimum; hence

$$\frac{du}{d\theta} = \sin (\beta - \alpha) \cos \theta - 2 \sin \beta \sin \alpha \sin \theta = 0;$$

and therefore, for equilibrium,

$$\tan \theta = \frac{\sin (\beta - \alpha)}{2 \sin \beta \sin \alpha},$$

a positive quantity, if, as we will suppose, β be greater than α.

Differentiating u a second time, we have

$$\frac{d^2 u}{d\theta^2} = - \sin (\beta - \alpha) \sin \theta - 2 \sin \beta \sin \alpha \cos \theta;$$

from this it appears that, since θ is clearly less than $\frac{\pi}{2}$, $\frac{d^2 u}{d\theta^2}$ will be negative, or that in the position of equilibrium the centre of gravity is at its maximum altitude; hence the equilibrium will be unstable.

(4) A square board hangs in a vertical plane by a string, which passing over a smooth nail has its ends fastened to two points symmetrically situated in one edge of the board. To investigate the positions and circumstances of equilibrium.

Let G (fig. 106), be the centre of gravity of the board, KCL the string passing over the nail C and attached to the board at the points K, L; draw GH at right angles to KL, and let ACB be an indefinite horizontal line.

Let l = the length of the string, $KL = a$, c = the length of a

side of the square, θ = the inclination of HG to the vertical, $\frac{1}{2}u$ = the distance of G below AB.

Since $CK + CL = l$, the locus of C, relatively to KL, is an ellipse of which K, L, are the foci. Now conceive θ to be for the present invariable: then it is evident that H and therefore G will be the lower, the lower the highest point of the ellipse: the highest point must therefore coincide with C, and AB must accordingly be a tangent to the ellipse.

The distance of H from the tangent of the ellipse, the axis major of the ellipse being l and eccentricity $\frac{a}{l}$, is equal to

$$\tfrac{1}{2}(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}:$$

hence $$u = c\cos\theta + (l^2 - a^2\cos^2\theta)^{\frac{1}{2}}.$$

Differentiating we shall get

$$\frac{du}{d\theta} = \sin\theta\left\{\frac{a^2\cos\theta}{(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}} - c\right\},$$

$$\frac{d^2u}{d\theta^2} = \cos\theta\left\{\frac{a^2\cos\theta}{(l^2 - a^2\cos^2\theta)^{\frac{1}{2}}} - c\right\} - \frac{l^2a^2\sin^2\theta}{(l^2 - a^2\cos^2\theta)^{\frac{3}{2}}}.$$

Putting $\frac{du}{d\theta} = 0$, we shall obtain

$$\theta = 0 \text{ or } \cos\theta = \frac{cl}{a(a^2 + c^2)^{\frac{1}{2}}}.$$

If $$\theta = 0,\quad \frac{d^2u}{d\theta^2} = \frac{a^2}{(l^2 - a^2)^{\frac{1}{2}}} - c:$$

if $$\cos\theta = \frac{cl}{a(a^2 + c^2)^{\frac{1}{2}}},\quad \frac{d^2u}{d\theta^2} = -\frac{l^2c^2\sin^2\theta}{a^4\cos^2\theta} = \text{a negative quantity}.$$

Thus we see that, if l be less than $\frac{a}{c}(a^2 + c^2)^{\frac{1}{2}}$, there will be three positions of equilibrium, and, if it be greater, only one. In the former case

$$\frac{a^2}{(l^2 - a^2)^{\frac{1}{2}}} - c = \text{a positive quantity},$$

and therefore $\theta = 0$ corresponds to a position of unstable, and

$$\cos\theta = \frac{cl}{a\,(a^2+c^2)^{\frac{1}{2}}}$$

to two positions of stable equilibrium.

In the latter case

$$\frac{a^2}{(l^2-a^2)^{\frac{1}{2}}} - c = \text{a negative quantity},$$

and therefore $\theta = 0$ corresponds to a position of stable equilibrium.

(5) A uniform slender rod, acted on by gravity, is placed with its extremities against two planes (one horizontal and the other vertical), having at a point in their intersection an attractive force, varying inversely as the square of the distance, which at the centre of gravity of the rod is equal to half the force of gravity; to find the position of equilibrium of the rod, and to ascertain whether it is stable or unstable.

Let AB (fig. 107) be the rod, G its centre of gravity, P any point in it; join OP, O being the centre of attraction; draw PM at right angles to the horizontal line OA. Let $AG = a = BG$, $AP = s$, $OP = r$, $PM = y$, $\angle\, OAB = \theta$; let the mass of a unit of the rod's length be taken as the unit of mass.

Then the attraction on an element δs of the rod at P will be equal to $g\delta s$ vertically downwards, and to $\frac{1}{2}g\,\frac{a^2}{r^2}\,\delta s$ towards the centre O. Hence, adopting the notation which was employed above in the enunciation of Maupertuis' Principle,

$$\Pi = \delta^{-1}d^{-1}\left\{g\delta s\,dy + \tfrac{1}{2}g\,\frac{a^2}{r^2}\delta s\,dr\right\},$$

and therefore $\quad d\Pi = g\delta^{-1}\,(\delta s\,dy) - \frac{1}{2}a^2g\,d\delta^{-1}\,\frac{\delta s}{r}\,.$

Now, by the geometry,

$$y = s\sin\theta, \quad dy = s\cos\theta\,d\theta;$$

hence $\quad \delta^{-1}\,(\delta s\,dy) = \delta^{-1}\,(s\delta s\cos\theta\;d\theta) = \delta^{-1}\,(s\delta s)\cos\theta\;d\theta,$

and consequently, the limits of the integration being obviously 0, $2a$,

$$\delta^{-1}(\delta s\, dy) = 2a^2 \cos\theta\, d\theta.$$

Again, by the geometry, we see that

$$r^2 = s^2 - 4a\cos^2\theta\, s + 4a^2\cos^2\theta;$$

hence we have

$$\delta^{-1}\frac{\delta s}{r} = \delta^{-1}\frac{\delta s}{(s^2 - 4as\cos^2\theta + 4a^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= C + \log\{s - 2a\cos^2\theta + (s^2 - 4as\cos^2\theta + 4a^2\cos^2\theta)^{\frac{1}{2}}\},$$

and therefore, between the limits $s = 0$, $s = 2a$,

$$\delta^{-1}\frac{\delta s}{r} = \log\frac{\sin\theta\,(1+\sin\theta)}{\cos\theta\,(1-\cos\theta)} = \log\frac{\tan\frac{1}{4}(\pi+2\theta)}{\tan\frac{1}{2}\theta}.$$

Hence
$$d\delta^{-1}\frac{\delta s}{r} = \frac{\frac{1}{2}\sec^2\frac{1}{4}(\pi+2\theta)}{\tan\frac{1}{4}(\pi+2\theta)} - \frac{\frac{1}{2}\sec^2\frac{1}{2}\theta}{\tan\frac{1}{2}\theta}$$

$$= \frac{1}{\cos\theta} - \frac{1}{\sin\theta}.$$

Putting for $\delta^{-1}(\delta s\, dy)$ and $d\delta^{-1}\dfrac{\delta s}{r}$ their values in the expression for $d\Pi$, we get

$$\frac{d\Pi}{d\theta} = \frac{1}{2}a^2 g\left(-\frac{1}{\cos\theta} + \frac{1}{\sin\theta} + 4\cos\theta\right).$$

Now there will be equilibrium if Π have a maximum or a minimum value, and therefore if

$$\sin\theta - \cos\theta - 4\sin\theta\cos^2\theta = 0;$$

multiplying this equation by $\cos\theta + \sin\theta$, a quantity which cannot be equal to zero from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, we get

$$-\cos 2\theta - (1+\cos 2\theta)(1+\sin 2\theta - \cos 2\theta) = 0,$$

$$\cos 2\theta + \sin 2\theta + \sin 2\theta(\cos 2\theta + \sin 2\theta) = 0,$$

$$(1+\sin 2\theta)(\cos 2\theta + \sin 2\theta) = 0;$$

but it is evident that $1+\sin 2\theta$ cannot become zero for any value of θ from 0 to $\frac{1}{2}\pi$; hence

$$\cos 2\theta + \sin 2\theta = 0, \quad \tan 2\theta = -1, \quad \theta = \tfrac{3}{8}\pi,$$

which determines the position of equilibrium.

Again, differentiating the expression for $\dfrac{d\Pi}{d\theta}$, we have

$$\frac{d^2\Pi}{d\theta^2} = -\tfrac{1}{2}a^2 g\left(\frac{\sin\theta}{\cos^2\theta} + \frac{\cos\theta}{\sin^2\theta} + 4\sin\theta\right),$$

which is evidently a negative quantity when $\theta = \frac{3}{8}\pi$; hence, for this value of θ, Π receives a maximum value, and therefore the equilibrium is one of instability.

(6) A particle is placed in a position of equilibrium between two centres of attractive force, varying according to any power of the distance; to determine for what laws of force the equilibrium is stable and for what unstable.

The equilibrium will be stable or unstable according as the forces attract in direct or inverse powers respectively.

(7) Two heavy particles, connected together by a thread PAQ (fig. 108) passing over the convex side of a circle situated in a vertical plane, balance each other when placed at P and Q; to determine the position of P, Q, and to ascertain whether the equilibrium is stable or unstable, the weight of the thread being neglected.

Let O be the centre of the circle, OA a vertical radius; let $\angle POQ = \alpha$, $\angle POA = \theta$, $\angle QOA = \phi$; and let m, n, denote the masses of the particles. Then we shall have for the equilibrium, which will be unstable, the equation

$$\tan\tfrac{1}{2}(\phi - \theta) = \frac{m-n}{m+n}\tan\tfrac{1}{2}\alpha.$$

(8) A uniform rod passes through a hole in a spherical shell, and rests with one end against the internal surface, the length of the rod being equal to twice that of the diameter; having given the inclination of the rod to the vertical when it is in a position of stable equilibrium, to determine its inclinations to the vertical when in its positions of unstable equilibrium.

If α denote its inclination to the vertical when in its position of stable equilibrium, then its inclinations for its two positions of unstable equilibrium will be

$$\tfrac{1}{3}(\pi + \alpha) \text{ and } \tfrac{1}{3}(\pi - \alpha).$$

(9) A board in the form of an isosceles triangle PQR (fig. 109), of which PQ is the base, is placed upon two inclined planes AL, AM, at right angles to each other, the plane of the triangle being vertical and perpendicular to the intersection of the two inclined planes: to find the position of equilibrium and to determine whether it is stable or unstable.

If $PQ = 2a$, $h =$ the altitude of the triangle, $\alpha =$ the inclination of AP and $\theta =$ that of PQ to the horizon: then the equation

$$\tan\theta = \frac{a\cos 2\alpha}{a\sin 2\alpha + \frac{1}{3}h}$$

will define a position of unstable equilibrium.

(10) To find the position and nature of the equilibrium of a cube which rests between two smooth inclined planes, the edges in contact with the planes being parallel to the line of their intersection.

If α, α', denote the inclinations of the two planes, and θ the inclination of the base of the cube to the horizon, the position of equilibrium, which is unstable, is given by the equation

$$\tan\theta = \frac{\sin(\alpha' - \alpha)}{\sin(\alpha + \alpha') + 2\sin\alpha\sin\alpha'}.$$

(11) A very small bar of matter is moveable about one extremity which is fixed half way between two centres of force attracting inversely as the square of the distance: to find the positions of the equilibrium of the bar and to determine their nature.

Let A, B, be the two centres of force, C the middle point between them, CL the position of the bar at rest. Let $AB = 2a$, $CL = l$, $\angle BCL = \phi$, and let μ, μ', denote the absolute forces of A, B, respectively.

Of the two quantities μ, μ', let μ be not the smaller: then, if

$$\frac{\mu}{\mu'} > \frac{a+2l}{a-2l} \text{..........................(1)},$$

there will be only two positions of equilibrium, defined by $\phi = 0$, $\phi = \pi$, the former unstable, the latter stable.

If the inequality (1) be not satisfied, $\phi = 0$, $\phi = \pi$, correspond to two positions of stable equilibrium; two unstable positions being given by the equation

$$\cos\phi = \frac{\mu - \mu'}{\mu + \mu'} \cdot \frac{a}{2l}.$$

(12) A heavy uniform rod AB hangs vertically downwards from a smooth hinge at A: each particle of the rod is attracted towards a centre of force at a point C, at a vertical distance above A equal to AB, according to the law of the first power of the distance: to ascertain the condition for stability or instability.

Let μ denote the absolute force, a the length AC or AB; then, if $\mu a < g$, the equilibrium is stable, and, if $\mu a > g$, unstable.

CHAPTER VII.

ATTRACTIONS.

(1) To find the attraction of the solid generated by the revolution of the curve $r^2 = a^2 \cos\theta$ round its axis, on a particle placed at the origin, the particles attracting inversely as the squares of the distances.

The required attraction, μ denoting the absolute force, is equal to

$$\int_0^{\frac{1}{2}\pi}\int_0^r r\,d\theta\,dr\,.\,2\pi r\sin\theta\,.\,\frac{\mu}{r^2}\cos\theta$$

$$= 2\pi\mu\int_0^{\frac{1}{2}\pi}\int_0^r \sin\theta\cos\theta\,d\theta\,dr$$

$$= 2\pi\mu a\int_0^{\frac{1}{2}\pi}(\cos\theta)^{\frac{3}{2}}\sin\theta\,d\theta$$

$$= -\,2\pi\mu a\left\{\frac{2}{5}(\cos\theta)^{\frac{5}{2}}\right\}_1^0$$

$$= \frac{4}{5}\pi\mu a.$$

(2) To determine that point in the axis of a hemispherical body, the particles of which attract inversely as the square of the distance, where a corpuscle must be placed so as to remain in equilibrium by the equal and contrary action of the matter of the hemisphere surrounding it.

Let CA (fig. 110) be the axis of the hemisphere, DCD' a diameter of its base, and O the required position of the corpuscle; DAD' the intersection of the plane through CA, DCD', with the surface of the hemisphere; draw BOB' at right angles to CA, join OD; take any points P, p, in the arcs AB, BD, join PO, po, and draw PM, pm, at right angles to CA. Let $CA = a = CD$, $CO = c$, $OB = b$, $OD = b'$, $OP = r$, $OM = x$,

$PM = y$, $Op = r'$, $Om = x'$, $pm = y'$; $\mu =$ the absolute attraction of a unit of mass of the hemisphere, and $\rho =$ its density; $A =$ the attraction of the portion BAB' of the hemisphere on the corpuscle, and B of the portion $BDB'D'$.

The attraction of a thin slice of the hemisphere at right angles to its axis at the point M, and having a thickness dx, will be

$$2\pi\mu\rho\, dx\left(1 - \frac{x}{r}\right),$$

as may be seen in elementary treatises on attraction; hence

$$A = 2\pi\mu\rho\int_0^{a-c} dx\left(1 - \frac{x}{r}\right),$$

$$\frac{A}{2\pi\mu\rho} = a - c - \int_0^{a-c} \frac{x\,dx}{r} \text{.......................} (1);$$

similarly we have

$$B = 2\pi\mu\rho\int_0^c dx'\left(1 - \frac{x'}{r'}\right),$$

$$\frac{B}{2\pi\mu\rho} = c - \int_0^c \frac{x'\,dx'}{r'} \text{ ....} (2).$$

Now from the geometry we see that

$$r^2 = x^2 + y^2 = x^2 + a^2 - (x + c)^2 = a^2 - c^2 - 2cx = b^2 - 2cx;$$

hence $\qquad 2cx = b^2 - r^2, \qquad c\,dx = -r\,dr,$

and therefore $\qquad \dfrac{x\,dx}{r} = -\dfrac{b^2 - r^2}{2c^2}\,dr;$

hence from (1), it being observed that r is equal to $a - c$, b, when x is equal to $a - c$, 0, we have

$$\frac{A}{2\pi\mu\rho} = a - c + \frac{1}{2c^2}\int_b^{a-c} (b^2 - r^2)\,dr \text{.............} (3).$$

Again, from the geometry,

$$r'^2 = x'^2 + y'^2 = x'^2 + a^2 - (c - x')^2 = a^2 - c^2 + 2cx' = b^2 + 2cx';$$

hence $\qquad 2cx' = r'^2 - b^2, \qquad cdx' = r'dr',$

and therefore $\qquad \dfrac{x'\,dx'}{r'} = -\dfrac{b^2 - r'^2}{2c^2}\,dr';$

hence from (2), since r' is equal to b', b, when x' is equal to c, 0,

$$\frac{B}{2\pi\mu\rho} = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r'^2)\, dr';$$

but it is evident that

$$\int_b^{b'} (b^2 - r'^2)\, dr' = \int_b^{b'} (b^2 - r^2)\, dr;$$

hence $$\frac{B}{2\pi\mu\rho} = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r^2)\, dr \quad\text{.............. (4).}$$

But, since the corpuscle is in equilibrium, we must have $A = B$, and therefore, by (3) and (4),

$$a - c + \frac{1}{2c^2}\int_b^{a-c} (b^2 - r^2)\, dr = c + \frac{1}{2c^2}\int_b^{b'} (b^2 - r^2)\, dr;$$

hence $$a - 2c = \frac{1}{2c^2}\int_{a-c}^{b'} (b^2 - r^2)\, dr:$$

performing the integration, and putting for b' its value $(a^2 + c^2)^{\frac{1}{2}}$, we shall get, after certain obvious simplifications,

$$a^3 - 4c^3 = (a^2 - 2c^2)\,(a^2 + c)^{\frac{1}{2}};$$

squaring both sides and simplifying,

$$12c^4 - 8a^3c + 3a^4 = 0,$$

an equation from which the value of c is to be determined; as an approximation $c = \frac{4}{9}a$.

Diarian Repository, p. 629.

(3) Two infinite lines in space, inclined to each other at a given angle, attract each other with forces varying inversely as the square of the distance: to find the whole attraction in the direction of the shortest line between them, the mutual attraction of two units of length collected in centres and separated by the unit of distance being considered equal to unity.

Let EE', FF', (fig. 111) be the two straight lines; AB the shortest distance between them. Take P, Q, any two points in the lines: join PQ. Through B draw a plane at right angles to AB, and let QK be the projection of QP on this plane.

Let $AB=c=PK$, $PQ=s$, $AP=r=BK$, $BQ=r'$, $\angle\, QPK=\phi$. Let $\theta=$ the angle between the two lines EE', FF'.

The mutual attraction of P and Q is equal to $\dfrac{dr\,.\,dr'}{s^2}$: and its resolved part parallel to AB is equal to

$$\frac{dr\,.\,dr'.\cos\phi}{s^2}=\frac{dr\,.\,dr'.\,s\cos\phi}{s^3}=\frac{dr\,.\,dr'.\,c}{s^3}.$$

But $$s^2=QK^2+PK^2=r^2+r'^2-2rr'\cos\theta+c^2.$$

Hence the whole mutual attraction parallel to AB is equal to

$$c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dr\,.\,dr'}{(c^2+r^2+r'^2-2rr'\cos\theta)^{\frac{3}{2}}}$$

$$=c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dr\,dr'}{\{(r'-r\cos\theta)^2+c^2+r^2\sin^2\theta\}^{\frac{3}{2}}}$$

$$=c\int_{-\infty}^{+\infty}\left[\frac{(r'-r\cos\theta)\,dr}{(c^2+r^2\sin^2\theta)\{(r'-r\cos\theta)^2+c^2+r^2\sin^2\theta\}^{\frac{1}{2}}}\right]_{-\infty}^{+\infty}$$

$$=2c\int_{-\infty}^{+\infty}\frac{dr}{c^2+r^2\sin^2\theta}=\frac{2}{\sin\theta}\left\{\tan^{-1}\frac{r\sin\theta}{c}\right\}_{-\infty}^{+\infty}=\frac{2\pi}{\sin\theta}.$$

(4) A slender ring DEF, (fig. 112), is attached to another slender ring ABC by means of a string AD, the length of which is equal to the radius of ABC; supposing DEF to fall entirely within ABC, to determine the tension of the string, when DEF is at rest: the force of attraction of ABC varying inversely as the cube of the distance.

It is plain that, when the smaller ring is at rest, AD will coincide with a radius of the larger.

Let P be any point in the smaller ring, R in the larger. Join PR, DR, DP, and produce DP to R. Let $a=$ the radius of the larger ring, $DE=a'$, $DP=c$, $PR=\rho$, $\angle\, PDC=\theta$, $\angle\, RDQ=\phi$, $\angle\, RPQ=\psi$; let ds, ds', be elements of the two rings at R, P, respectively, and μ, μ', the masses of units of length of the two rings; let $T=$ the required tension.

Then $$T = \mu\mu' \int \left\{ \cos\theta\, ds' \int \frac{ds}{\rho^3} \cos\psi \right\}.$$

Now $$ds = a\,d\phi, \quad ds' = \tfrac{1}{2} a'\, d(\pi - 2\theta) = -a'\,d\theta.$$

Hence $$T = \mu\mu' a a' \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \left\{ \cos\theta\, d\theta \int_0^{2\pi} \frac{d\phi}{\rho^3} \cos\psi \right\}.$$

$$\int_0^{2\pi} \frac{d\phi}{\rho^3} \cos\psi = \int_0^{2\pi} \frac{a\cos\phi - c}{(a^2+c^2-2ac\cos\phi)^{\frac{3}{2}}}\, d\phi = \frac{d}{dc}\int_0^{2\pi} \frac{d\phi}{a^2+c^2-2ac\cos\phi}$$

$$= \frac{d}{dc}\int_0^{2\pi} \frac{\sec^2\frac{\phi}{2}\, d\phi}{(a^2+c^2)\left(1+\tan^2\frac{\phi}{2}\right) - 2ac\left(1-\tan^2\frac{\phi}{2}\right)}$$

$$= 2\frac{d}{dc}\int_0^{2\pi} \frac{d\tan\frac{\phi}{2}}{(a-c)^2 + (a+c)^2 \tan^2\frac{\phi}{2}}$$

$$= 2\frac{d}{dc}\,{}_0^{2\pi}\left\{\frac{1}{a^2-c^2}\cdot \tan^{-1}\left(\frac{a+c}{a-c}\tan\frac{\phi}{2}\right)\right\}$$

$$= 2\frac{d}{dc}\left(\frac{\pi}{a^2-c^2}\right) = \frac{4\pi c}{(a^2-c^2)^2}.$$

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \cos\theta\, d\theta \,.\, \frac{4\pi c}{(a^2-c^2)^2} = 4\pi a' \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{\cos^2\theta\, d\theta}{(a^2 - a'^2\cos^2\theta)^2}$$

$$= 4\pi a' \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{\sec^2\theta\, d\theta}{(a^2 - a'^2 + a^2\tan^2\theta)^2}$$

$$= \frac{4\pi a'}{a}\int_{-\infty}^{+\infty} \frac{dx}{(a^2 - a'^2 + x^2)^2} = \frac{4\pi^2 a'}{a\,(a^2-a'^2)^{\frac{3}{2}}}.$$

Hence $$T = \frac{4\pi^2 \mu\mu' a'^2}{(a^2 - a'^2)^{\frac{3}{2}}}.$$

(5) To determine how much of the Earth's surface, considered spherical, a person can see, who is raised to such a height as to lose $\left(\frac{1}{n}\right)^{\text{th}}$ part of his weight.

If r = the radius of the Earth, the visible area is equal to

$$2\pi r^2 \left\{1 - \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\right\}.$$

(6) ACC' is a thin lamina, bounded by CAC', an arc of a lemniscate, (the node of which is O and equation $r^2 = a^2 \cos 2\theta$), and CC', a circular arc, of which O is the centre and radius $a \sin \epsilon$. To find the law of the variation of the resultant attraction of the lamina upon a molecule at O, when ϵ varies; the particles of the lamina being supposed to attract according to the law of nature.

The resultant attraction varies as

$$\log\left(\cot\frac{\epsilon}{2}\right) - \cos\epsilon.$$

(7) The sides of an isosceles triangle are formed of slender uniform prisms, attracting with forces which vary inversely as the square of the distance; to determine the vertical angle in order that a particle may remain at rest in a point which divides the perpendicular from the vertex in a given ratio.

If a be the distance of the particle from the vertex, and b from the base, then

$$\text{the vertical angle} = 2\sin^{-1}\left(\frac{b}{a}\right).$$

(8) Two straight lines AB, AC, at right angles to one another, attract a particle P placed at the point where the perpendicular AP meets BC; to find the direction and magnitude of the force necessary to keep the particle at rest, the law of attraction being that of the inverse square.

Let $AB = a$, $AC = b$, $BC = c$, μ = the absolute force of a unit of length of the attracting lines condensed into a point; then the direction of the required force will make an angle of 45° with AB, and its magnitude will be equal to

$$\frac{\sqrt{(2)}\,\mu c^3}{a^2 b^2}.$$

(9) A particle is attached, by means of a fine string, to the centre of a thin hemispherical shell of attractive matter; to determine the tension of the string, supposing its length to be less than the radius of the shell, the force of attraction varying inversely as the square of the distance.

If r = the radius of the shell, c = the length of the string, μ = the attraction of a unit of the shell's mass condensed at a unit of distance, τ = the thickness of the shell, the tension will be equal to

$$\frac{2\pi\mu\tau r^2}{c^2}\left\{1-\frac{r}{(c^2+r^2)}\right\}.$$

(10) A molecule is placed at a point within a triangle ABC, formed of three uniform rods of equal thickness, which attract according to the law of the inverse square, the densities of the rods BC, CA, AB, being λ, μ, ν, respectively: to find the conditions for the equilibrium of the particle.

If p, q, r, be the perpendicular distances of the molecule from BC, CA, AB, respectively, then

$$\frac{p}{\lambda}=\frac{q}{\mu}=\frac{r}{\nu}.$$

If $\lambda=\mu=\nu$, then $p=q=r$, or the molecule will rest at the centre of the inscribed circle, a theorem proved by Ferdinand Joachimsthal, in the *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 93.

(11) Two equal straight rods, the particles of which attract each other with a force varying inversely as the square of the distance, are parallel to each other and perpendicular to the lines joining their ends, and are held asunder by strings attached to their middle points: to determine the tension of the strings when the rods are at a given distance from each other.

If a = the distance between the rods and b = the length of either, the required tension is equal to

$$\frac{2\mu}{a}\{(a^2+b^2)^{\frac{1}{2}}-a\}.$$

(12) A brittle rod AB, attached to smooth hinges at A and B, is attracted towards a centre of force C according to the law of nature. Supposing the absolute force to be indefinitely augmented, to determine where the rod will eventually snap.

If E be the point of snapping, then, α, β, denoting the angles BAC, ABC, respectively,

$$\cos \angle AEC = \frac{\sin \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\beta}{2}}.$$

Mackenzie and Walton's *Solutions of the Cambridge Problems for* 1854.

CHAPTER VIII.

MISCELLANEOUS PROBLEMS.

(1) STRINGS are fastened to any number of points A, B, C, ..., in space, and are pulled towards a point P with forces proportional to PA, PB, PC, ...: shew that, wherever the point P be situated, the resultant of these forces will always pass through a fixed point.

Let a, b, c, be the co-ordinates of P referred to three rectangular axes: then, x, y, z, being the co-ordinates of any one of the points A, B, C, ..., the components of the resultant will be equal to

$$\mu(na - \Sigma x), \quad \mu(nb - \Sigma y), \quad \mu(nc - \Sigma z),$$

which will therefore be proportional to the direction-cosines of the resultant. The equations to the resultant will therefore be

$$\frac{x' - a}{na - \Sigma x} = \frac{y' - b}{nb - \Sigma y} = \frac{z' - c}{nc - \Sigma z}:$$

multiplying each of these fractions by n and adding unity to each we get

$$\frac{nx' - \Sigma x}{na - \Sigma x} = \frac{ny' - \Sigma y}{nb - \Sigma y} = \frac{nz' - \Sigma z}{nc - \Sigma z}.$$

Hence we see that the resultant always passes through a point of which the co-ordinates are

$$\frac{1}{n}\Sigma x, \quad \frac{1}{n}\Sigma y, \quad \frac{1}{n}\Sigma z.$$

(2) Find the amount of *work done* in drawing up a common Venetian blind. How must the same problem be solved for a curtain?

Let W = the weight of each bar of the blind; a = the distance between two consecutive bars; n = their number. Then the work done will be equal to

$$W(a + 2a + 3a + \dots\dots + na)$$
$$= \tfrac{1}{2} n(n+1)\, Wa.$$

Let P = the sum of the weights and l = the height of the window: then $P = nW$ and $l = na$, and the work done is equal to

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) Pl.$$

Let $n = \infty$: then the Venetian blind is mechanically the same as a curtain, the number of its bars being infinite and the weight of each indefinitely small. Thus, P being the weight and l the length of the curtain, the work done is equal to

$$\tfrac{1}{2} Pl.$$

The work done in raising the curtain may also be estimated by integration.

The weight of a length dx of the curtain is $\frac{P}{l}\,dx$: hence the work done

$$= \int_0^l \frac{P}{l}\, dx \,.\, x = \tfrac{1}{2} Pl.$$

(3) The frustum of a paraboloid of revolution, the density of its circular sections varying as their areas, stands upon its vertex on a horizontal plane: to find the length of its axis when the equilibrium is indifferent.

If the vertex of a solid of revolution, of which the axis extends vertically upwards, be placed upon the summit of another solid of revolution the axis of which extends vertically downwards, then, as is proved in most works on Statics, the equilibrium will be stable, unstable, or indifferent, accordingly as the altitude of the centre of gravity of the upper body above the point of contact is less than, greater than, or equal to

$$\frac{rr'}{r + r'},$$

r, r', being the radii of curvature of the two surfaces at the point of contact.

If r', the radius of curvature of the lower surface, be equal to ∞, the lower surface becomes a plane, and the expression

$$\frac{rr'}{r+r'}, \text{ becomes } r.$$

In the present question, as we may easily ascertain, the altitude of the centre of gravity is equal to $\frac{3}{4}c$, c being the length of the axis. Also the radius of curvature at the vertex is equal to $\frac{1}{2}l$, l being the latus rectum. Hence

$$\tfrac{3}{4}c = \tfrac{1}{2}l, \qquad c = \tfrac{2}{3}l.$$

(4) A beam can turn in every direction about one end which is fixed. The other rests on the upper surface of a rough plane, (the coefficient of friction being μ), which is inclined to the horizon at an angle α. If β be the angle the beam makes with the plane, prove that the beam will rest in any position if $\tan\alpha$ be not greater than

$$\frac{\mu}{(1+\mu^2\tan^2\beta)^{\frac{1}{2}}}.$$

Let O, (fig. 113), be the fixed end; OC a perpendicular upon the rough plane; CB a section of the rough plane by a vertical plane through OC; OER a horizontal line cutting BC in E; the circular quadrantal arc APB the locus of the free end of the beam; P the place of the end of the beam for a limiting position of equilibrium; PQ a line parallel to AC, and QR at right angles to OER.

Let $l=$ the length of the beam, $\angle PCQ=\theta$, $R=$ the normal reaction of the rough plane at P; then, the horizontal component of R being $R\sin\alpha$, parallel to EO, and the horizontal components of μR being $\mu R\cos\theta$ along PQ and $\mu R\sin\theta\cos\alpha$ parallel to EO, we have for the equilibrium of the beam, taking moments about a vertical line through O,

$$R\sin\alpha\,.\,PQ=\mu R\cos\theta\,.\,OR+\mu R\sin\theta\cos\alpha\,.\,PQ,$$

$$\sin\alpha\,.\,l\cos\beta\sin\theta$$

$$=\mu\cos\theta\left\{l\frac{\sin\beta}{\sin\alpha}+(l\cos\beta\cos\theta-l\sin\beta\cot\alpha)\cos\alpha\right\}$$

$$+\mu\sin\theta\cos\alpha\,.\,l\cos\beta\sin\theta,$$

$$\sin\alpha\cos\beta\sin\theta = \mu\sin\beta\cos\theta\sin\alpha + \mu\cos\alpha\cos\beta,$$

$$\tan\alpha\tan\theta - \mu\tan\alpha\tan\beta = \mu(1+\tan^2\theta)^{\frac{1}{2}},$$

$$(\tan^2\alpha - \mu^2)\tan^2\theta - 2\mu\tan^2\alpha\tan\beta\tan\theta + \mu^2(\tan^2\alpha\tan^2\beta - 1) = 0.$$

That $\tan\theta$ may be impossible, or that there may be no *limiting* equilibrium, we must have

$$\tan^4\alpha\tan^2\beta < (\tan^2\alpha - \mu^2)(\tan^2\alpha\tan^2\beta - 1),$$

or
$$\tan^2\alpha(1+\mu^2\tan^2\beta) < \mu^2.$$

A different solution of this problem may be seen in the *Solutions of the Senate-House Problems* for 1844, by O'Brien and Ellis.

(5) A system consists of n equal particles which have no initial velocities: prove that it will remain at rest, if their co-ordinates can only vary subject to the condition

$$n\Sigma(x^2+y^2+z^2) - (\Sigma x)^2 - (\Sigma y)^2 - (\Sigma z)^2 = \text{a constant}:$$

the particles attracting one another with a force which varies as the distance.

The attraction between any two of the particles P_1, P_2, at a distance r from each other is proportional to r. Conceive the system to experience any indefinitely small displacement consistently with its geometrical connexions, and let α denote the component of P_1's motion estimated along P_1P_2, and β that of P_2's motion estimated along P_2P_1. Then $(\alpha+\beta)r$ will denote the sum of the *moments* of the two forces; but $(\alpha+\beta) = -dr$: hence, by the principle of Virtual Velocities, equating to zero the sum of the *moments* of the whole system, we have

$$0 = \Sigma(r\,dr),$$

whence
$$\begin{aligned} C &= 2\int\Sigma(rdr) \\ &= \Sigma(r^2) \\ &= \Sigma\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\} \\ &= (n-1)\Sigma(x^2+y^2+z^2) - 2\Sigma(xx'+yy'+zz') \\ &= n\Sigma(x^2+y^2+z^2) - (\Sigma x)^2 - (\Sigma y)^2 - (\Sigma z)^2. \end{aligned}$$

(6) If an elliptic board be placed, so that its plane is vertical, on two pegs which are in a horizontal line, there will be equilibrium if these pegs be at the extremities of a pair of conjugate diameters. What are the limits which the distance between the pegs must not exceed or fall short of, in order that this position of equilibrium may be possible? Shew that the position is one of unstable equilibrium.

Let P', P'', (fig. 114), be the two pegs: C the centre of the ellipse, CA the semi-axis major: draw $P'M'$, $P''M''$, at right angles to CA and CQ at right angles to $P'P''$.

Let $P'P''=c$, $CQ=u$, $CM'=x'$, $P'M'=y'$, $CM''=x''$, $PM''=y''$.

Then, equating CQ to the difference of the projections of CM', $P'M'$, upon its direction, we get

$$u=\frac{x'+x''}{c}y'-\frac{y'-y''}{c}x'=\frac{x''y'+x'y''}{c}.$$

Put $x'=a\cos\phi'$, $y'=b\sin\phi'$, $x''=a\cos\phi''$, $y''=b\sin\phi''$, a and b being the semi-axes of the ellipse: then, if $\phi'+\phi''=\psi$,

$$cu=ab\sin(\phi'+\phi'')=ab\sin\psi.$$

That u may be a maximum or minimum,

$$c\frac{du}{d\psi}=ab\cos\psi=0,$$

whence $\psi=\frac{1}{2}\pi$, which shews, by a known property of the ellipse, that P', P'', are extremities of conjugate diameters of the ellipse.

Again,

$$c\frac{d^2u}{d\psi^2}=-ab\sin\psi=-ab:$$

hence u is a maximum, or the equilibrium is unstable.

Moreover,
$$\begin{aligned} c^2 &= (x'+x'')^2+(y'-y'')^2 \\ &= a^2(\cos\phi'+\cos\phi'')^2+b^2(\sin\phi'-\sin\phi'')^2 \\ &= a^2(\cos\phi'+\sin\phi')^2+b^2(\sin\phi'-\cos\phi')^2 \\ &= a^2+b^2+(a^2-b^2)\sin 2\phi'. \end{aligned}$$

Hence we see that the greatest and least limits of c are $a\sqrt{2}$ and $b\sqrt{2}$.

(7) A flexible thread rests upon a smooth surface, under the action of any forces: to investigate its form.

Let $R =$ the reaction of the surface at any point (x, y, z) of the thread, t being the tension at that point. Then, for the equilibrium of the thread, λ denoting a certain coefficient, and $m\delta s$ the mass of an element δs of the thread,

$$\frac{d}{ds}\left(t\frac{dx}{ds}\right) + mX + \lambda R\frac{du}{dx} = 0 \text{.................} (1),$$

$$\frac{d}{ds}\left(t\frac{dy}{ds}\right) + mY + \lambda R\frac{du}{dy} = 0 \text{.................} (2),$$

$$\frac{d}{ds}\left(\lambda\frac{dz}{ds}\right) + mZ + \lambda R\frac{du}{dz} = 0 \text{.................} (3).$$

Also, if $u = 0$ be the equation to the surface,

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0 \text{.................} (4).$$

From (1), (2), (3), eliminating $\frac{dt}{ds}$ and λR by cross multiplication, we get

$$0 = \left(mX + t\frac{d^2x}{ds^2}\right)\left(\frac{dy}{ds}\frac{du}{dz} - \frac{dz}{ds}\frac{du}{dy}\right) + \left(mY + t\frac{d^2y}{ds^2}\right)\left(\frac{dz}{ds}\frac{du}{dx} - \frac{dx}{ds}\frac{du}{dz}\right)$$
$$+ \left(mZ + t\frac{d^2z}{ds^2}\right)\left(\frac{dx}{ds}\frac{du}{dy} - \frac{dy}{ds}\frac{du}{dx}\right) \text{.............} (5).$$

But, from (1), (2), (3), (4),

$$0 = dx\frac{d}{ds}\left(t\frac{dx}{ds}\right) + dy\frac{d}{ds}\left(t\frac{dy}{ds}\right) + dz\frac{d}{ds}\left(t\frac{dz}{ds}\right)$$
$$+ m(Xdx + Ydy + Zdz),$$

and therefore

$$0 = dt + m(Xdx + Ydy + Zdz) \text{................} (6).$$

The equations (5), (6), together with the equation to the surface, determine the form of the thread.

(8) From a square $ABCD$ a triangle AEF is cut, AE being the fourth part of AD, and AF three-fourths of AB; to find the centre of gravity of the remaining figure $BCDEF$.

If a denote a side of the square, the distances of the required centre of gravity from AB, AD, respectively, are

$$\frac{63}{116}a, \qquad \frac{61}{116}a.$$

(9) The points D, E, F, divide the sides BC, CA, AB, of a triangle proportionally, that is, so that

$$BD : CE : AF :: DC : EA : FB;$$

shew that the centre of gravity of the triangle DEF coincides with that of the triangle ABC.

(10) The diagonals of a trapezium intersect at right angles in a fixed point, and have always the same directions, the magnitudes of the diagonals and of one side being given: prove that the locus of the centre of gravity of the trapezium is a circle of which the radius is $\frac{1}{3}c$.

(11) A triangular plate hangs by three parallel threads attached at the corners, and supports a heavy particle. Prove that, if the threads are of equal strength, a heavier particle may be supported at the centre of gravity than at any other point of the disk.

(12) Two forces in the ratio of $1+n$ to 1, where n is small, act upon a point in directions including an angle α; shew that the sine of the angle which the direction of the resultant makes with that of the larger force is nearly equal to

$$(1-\tfrac{1}{2}n)\sin\frac{\alpha}{2}.$$

(13) If three forces, represented in magnitude and direction by lines OA, OB, OC, act at a point O, not necessarily in the plane of ABC, prove that their resultant will be represented in magnitude and direction by $3OG$, G being the centre of gravity of the triangle ABC.

(14) Forces represented by $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, act at the angular points

of a triangle ABC, right-angled at C, in the directions of the sides taken in order; prove that the resultant is represented by

$$\left(\frac{c^4}{a^2b^2} - \frac{3}{c^2}\right)^{\frac{1}{2}},$$

that it is inclined to AC at an angle $\cos^{-1} \frac{a^3}{(a^6+b^6)^{\frac{1}{2}}}$, and that it cuts BC produced at a distance $\frac{b^2}{a}$ from C.

(15) Prove that, if, at each point of space, a force act which is any function of its distance from a given point A, and θ be the angle at which the tangent at a point P of an arbitrary curve, connecting any two points P_1, P_2, in space, is inclined to the direction of the force f at P, then $\int f \cos\theta ds$ from P_1 to P_2 depends only on the distances AP_1, AP_2.

(16) Eight centres of force, acting in the corners of a cube, attract, according to the same law and with the same absolute intensity, a particle placed very near the centre of the cube: shew that their resultant action passes through the centre of the cube, unless the law of force be that of the inverse square of the distance.

(17) Shew that a system of forces acting in one plane, and represented by the sides of a polygon, is equivalent to a couple the moment of which is represented by twice the area of the polygon.

(18) Four unequal forces P, Q, R, S, act upon a rigid body along the sides OA, AB, BC, CO, of a square $OABCO$: prove that, OA and OC being taken as the axes of co-ordinates, there will be a single resultant force the equation to which, if a be a side of the square, is

$$(Q-S)x + (R-P)y = a(Q+R),$$

and of which the magnitude is

$$\{(P-R)^2 + (Q-S)^2\}^{\frac{1}{2}}.$$

(19) Three forces act in equilibrium at the angles of a triangle, one bisecting the angle at which it acts, and the other two making equal angles with the side opposite to that angle; shew that the forces are as the sides opposite to their points of application.

(20) Assuming friction to consist of the sum of two parts, the one proportional to the pressure, and the other to the surface in contact, shew that, when a parallelepiped, the edges of which are a, b, c, is supported with one edge parallel to the horizon on a given inclined plane by the least possible force acting in a given direction through its centre of gravity at right angles to this edge, we shall have

$$\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0;$$

p, q, r, being the values of the force in question, when the parallelepiped rests on the faces bc, ca, ab, respectively.

(21) A triangular disk, the sides of which are a, b, c, is suspended from a fixed point by threads α, β, γ, attached to its corners, α being the length of the thread attached to the corner opposite to a, and so of the rest. If the plane of the disk be horizontal, prove that

$$a^2+3\alpha^2=b^2+3\beta^2=c^2+3\gamma^2.$$

(22) Three equal heavy rods, in the position of the three edges of an inverted triangular pyramid, are in equilibrium under the following circumstances: their upper extremities are connected by fine strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Shew that the increase of tension of the strings, corresponding to a given small increase of their lengths, varies inversely as $\sin^3\theta$, where θ is the inclination of each of the rods to the horizon.

(23) A cone, the density of the circular sections of which varies as their distances from its vertex, will balance on the middle point of its axis, if a weight equal to three-fifths of its own weight be suspended at the vertex.

(24) A solid, generated by the revolution of a semicircle round its diameter through an angle of 60°, is placed upon a smooth horizontal plane; determine the moment of the couple which will keep the axis of revolution of the solid in a vertical position.

If $a =$ the radius of the circle, and $\rho =$ the density of the material, the required moment is equal to $\frac{1}{8} g\rho\pi a^4$.

(25) If the frustum of a cone be bisected by a plane through its axis, prove that either half will just stand upon the smaller end on a horizontal plane, if

$$\frac{h+h'}{h} = \pi \frac{h^2 + hh' + h'^2}{h^2 + h'^2},$$

where h, h', are the heights of the smaller and larger cones the difference of which constitutes the frustum.

(26) Two spheres, attached to the two ends of a fine string, which hangs over a fixed point, rest in contact: prove that their weights are inversely as the distances of their centres from the point of suspension.

(27) A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it: if each projects as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

(28) A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting upon a horizontal plane: all the blocks above the lowest are then twisted in the same direction about an edge of the column, first the highest, then the two highest, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines.

(29) A quadrilateral is formed by four rigid rods jointed at the ends; shew that two of its sides must be parallel in order

that it may preserve its form when the middle points of either pair of opposite sides are joined together by a string in a state of tension.

(30) Any sector of a circle is placed with its lower radius horizontal, the plane of the sector being vertical: shew that a heavy uniform chain laid over the arc and upper radius, so as just to coincide with them in every part, will remain in equilibrium.

(31) The ends of a uniform chain are fastened to two fixed points A and B in a horizontal line: a link C of the chain rests upon a rigid wire which joins A and B, so that the chain forms two festoons AC and CB. Prove that, if there be no friction between the wire and chain, the smaller of these festoons is equal and similar to a portion of the larger.

(32) A uniform chain of length $2l$ is suspended from two points in a horizontal line; if $2a$ be the distance between the points of support, t, c, the respective lengths of the chain the weights of which are equal to the tensions at either point of support and at the lowest point of the chain, prove that, when l has such a value that t is a minimum,

$$ct = al.$$

(33) To determine the form in which a chain will hang, suspended at two points, when the density at any point varies as the tension at that point; the thickness of the chain being uniform.

The axis of x being horizontal and that of y vertical, and the origin being at the lowest point, the equation to the curve will be

$$e^{\frac{y}{c}} = \sec \frac{x}{c},$$

c being a constant.

(34) A cone rests with its base upon the vertex of a given paraboloid: prove that, for stability of equilibrium, it is necessary that the height of the cone be less than twice the latus rectum of the paraboloid.

(35) If a cone of the same substance and of equal base with a hemisphere be fixed to the latter, so that their bases coincide, to find the greatest height of the cone in order that the equilibrium may be stable, when the hemisphere rests symmetrically on a horizontal plane.

The height of the cone must be less than $r\sqrt{3}$, r being the radius of the hemisphere.

(36) If a solid cylinder be cut by a plane which touches the circumference of its base at a point A and meets the axis at an angle of 45°, prove that the piece of the cylinder included between the cutting plane and the base will rest in indifferent equilibrium, if placed with its circular end on the vertex of a paraboloid the latus rectum of which is $\frac{3}{8}$ths the diameter of the base, the point of contact being also at this same distance from A.

DYNAMICS.

CHAPTER I.

IMPACT AND COLLISION. SMOOTH SPHERICAL BODIES.

CONCEIVE two spherical bodies, which are composed of the same material, to be moving in the same straight line, namely, in the line joining their centres, and at any time during their motion to impinge against each other. Let m, m', denote the masses of the two bodies; and let u, u', represent their velocities before and v, v', their velocities after collision; the symbols which represent the velocities being positive when motion takes place in one direction and negative when it takes place in the other; then, whatever be the magnitudes of u, u', or of m, m',

$$u - u' : v' - v :: 1 : e,$$

or

$$v' - v = e\,(u - u') \quad \text{......................(A)},$$

where e is a numerical quantity not greater than unity, which is invariable while the material of the bodies remains the same, but which changes generally with a change in their substance. The bodies are said to be inelastic if e be equal to zero; imperfectly elastic if it be equal to any fraction between zero and unity; and perfectly elastic if it be equal to unity.

The theory of collision furnishes us likewise with the following general relation,

$$m\,(u - v) = m'\,(v' - u') \quad \text{......................(B)}.$$

The equations (A) and (B) are usually more convenient when written in the forms

$$eu + v = eu' + v',$$

$$mu + m'u' = mv + m'v'.$$

The signification of the equation (A) is, that the relative velocity of the two bodies after collision bears a constant ratio to the relative velocity before collision, so long as the material of the bodies remains unchanged; and the equation (B) implies, that the momentum which one body gains by the collision in the positive direction of motion, is equal to that which the other loses. These are the two fundamental principles in the theory of collision.

Suppose that u' is equal to zero, and that m is inconsiderable in comparison with m'; then clearly, by (A) and (B),

$$v' - v = eu \text{ and } v' = u' = 0,$$

and therefore
$$v = -eu \qquad \text{(C)},$$

or the small body is reflected backwards with a velocity which is to the velocity of impact as e to 1; while the large body experiences no appreciable motion from the collision. This is evidently the case of bodies impinging and rebounding upon the surface of the earth, or upon other bodies firmly attached to it, the earth being regarded as stationary.

In the year 1639, J. Marc Marci de Crownland[1], a Hungarian physician, published at Prague a work entitled *De Proportione Motus, seu Regula Sphymica*, in which he has treated of the collision of perfectly elastic and perfectly inelastic bodies. He occupies himself principally with the consideration of perfectly elastic bodies, and lays down precisely the same rules for their collision which are now commonly adopted. This work, the earliest in which the theory of collision had been correctly propounded, having fallen into general oblivion in the scientific world, the subject was again correctly investigated by the independent efforts of Wallis, Wren, and Huyghens, who apparently had not the slightest knowledge even of the existence of the work by Marci. The laws of the collision of perfectly inelastic bodies were laid down by Wallis, *Phil. Trans.* 1668, p. 864, and of perfectly elastic bodies by Wren, *Phil. Trans.* 1668, p. 867, and Huyghens, *Phil. Trans.* 1669, p. 925, and *Journal des Sçavans* of March 18, 1669. Wren and Lawrence Rook

[1] Montucla; *Histoire des Mathématiques*, Tom. II. p. 406.

had, several years earlier than this, exhibited various experiments before the Royal Society, in illustration of the principles of collision. The conclusions of Wallis, Wren, and Huyghens, which had been presented to the Royal Society in a very brief shape, were afterwards given more at large by Wallis, *Mechanica, Pars Tertia*, 1671; Keill, *Introductio ad Veram Physicam*, Lect. 12, 13, 14; and Mariotte, *Traité de Percussion*. There are some ingenious experiments by Smeaton on the theory of collision in the *Phil. Trans.*, April 18, 1782. The principles of the collision of imperfectly elastic bodies were first propounded by Newton, *Principia*, Lib. I., Scholium to the Laws of Motion, who inferred experimentally the truth of the equation (A) for any value whatever of e between zero and unity; preceding philosophers having directed their attention to those cases alone in which e is supposed to be either zero or unity. The physical value of Newton's generalization is the more striking when it is considered that natural bodies are never actually endowed with perfect elasticity. For the mathematical formulæ in the theory of the collision of imperfectly elastic spheres, the reader is referred to Maclaurin, *Choc des Corps, Prix de l'Academie*, Tom. I. and to Bossut, *Cours de Mathématique*, Tom. III. The results of a series of experiments on the elasticity of bodies, by Mr Hodgkinson, are to be found in Vol. III. p. 534, of the *Reports of the British Association for the Advancement of Science*, where he has shewn that the quantity e, in the equation (A), is not, as we stated, and as we shall suppose for the sake of mathematical simplicity, entirely independent of the velocities of the impinging bodies, as Newton had concluded, but that it decreases as the relative velocity increases, assuming however a nearly constant value when the relative velocity of collision becomes considerable.

(1) Two inelastic bodies are moving in opposite directions with given velocities; to find their velocities after collision.

Let m, m', denote the masses of the bodies; a, a', their velocities before collision. Then, putting in the formulæ (A) and (B),

$$u = a, \quad u' = -a', \quad e = 0,$$

we have $$v' - v = 0, \quad m(a - v) = m'(v' + a'),$$

and therefore $$m(a-v) = m'(a'+v),$$

$$v' = v = \frac{ma - m'a'}{m+m'}.$$

If then ma be greater than $m'a'$, the bodies will, after collision, move along, in the direction in which m originally moved, with a common velocity $(ma - m'a') : (m + m')$; and, if ma be less than $m'a'$, they will move in the opposite direction with a common velocity $(m'a' - ma) : (m' + m)$. If ma be equal to $m'a'$, the collision will reduce both the bodies to rest.

Wallis; *Mechan. Pars Tertia, de Percussione*, Prop. IV.

(2) Two perfectly elastic bodies are moving in opposite directions with given velocities; to find their velocities after collision.

The notation being the same as in the preceding example, we have, e being in this case equal to unity,

$$v' - v = a + a', \quad m(a-v) = m'(a'+v').$$

Eliminating v' from these two equations,

$$v = \frac{ma - m'a}{m+m'} - \frac{2m'a'}{m+m'}.$$

Eliminating v, we have

$$v' = \frac{ma' - m'a'}{m+m'} + \frac{2ma}{m+m'}.$$

Wallis; *Ib. de Elatere et Resilitione*, Prop. X.

(3) Three bodies m, m', m'', are placed in a row. The body m receiving a given velocity towards m', to find the magnitude of m' that the velocity communicated to m'' by its intervention may be the greatest possible.

Let a be the velocity with which m is projected; a' the velocity which m' acquires on being struck by m, and a'' that which m'' receives on being struck by m'. Then

$$a' = \frac{2ma}{m+m'}, \quad a'' = \frac{2m'a'}{m'+m''},$$

and therefore $$a'' = \frac{4mm'a}{(m+m')(m'+m'')}.$$

Since a'' is to be a maximum, we must have

$$\left(\frac{m}{m'}+1\right)\left(m'+m''\right) \text{ a minimum;}$$

hence, differentiating with respect to the variable m',

$$\frac{m}{m'}+1-\frac{m}{m'^2}(m'+m'')=0,$$

$$m'^2-mm''=0, \quad m'=(mm'')^{\frac{1}{2}}.$$

Huyghens; *Phil. Trans.* 1669, p. 928.
Wolff; *Elementa Matheseos Universæ*, Tom. II. p. 158.

(4) A perfectly elastic sphere impinges with a given velocity and in a given direction against a smooth plane; to determine the velocity and direction of reflection.

Let u, v, denote the velocities of incidence and of reflection, and α, β, the angles which the directions of the motion before and after impact make with a normal to the plane.

The resolved parts of the velocities parallel to the plane are $u \sin \alpha$ and $v \sin \beta$, and, at right angles to it, $u \cos \alpha$ and $v \cos \beta$. But the plane being perfectly smooth will not affect the resolved part of the velocity parallel to itself, and therefore

$$v \sin \beta = u \sin \alpha;$$

while the other resolved part of the incident velocity will be affected as if the impact had been direct, and therefore, by (C),

$$v \cos \beta = u \cos \alpha.$$

From these two equations it is evident that

$$\tan \beta = \tan \alpha, \quad \beta = \alpha, \quad \text{and } v = u,$$

or the angle of reflection is equal to that of incidence, and the velocity of reflection to the velocity of incidence.

Wallis; *Mechan. Pars Tertia, De Elatere*, &c. Prop. II.

(5) Two smooth spheres moving with given velocity and in any given directions whatever, impinge against each other; the spheres being supposed perfectly elastic, to determine their velocities and the directions of their motions after collision.

Let AB, $A'B'$, (fig. 115), be the directions of the motion of the two bodies before collision; and O, O', the positions of their centres at the instant of contact; produce $O'\ O$ indefinitely to a point C. Let a, a', denote their velocities before collision, and α, α', the angles AOC, $A'O'C$.

Then the resolved parts of the velocities of the spheres O, O', in the direction COO' will be $a\cos\alpha$, $a'\cos\alpha'$, and at right angles to OO' in the planes AOC, $A'O'C$, respectively, $a\sin\alpha$, $a'\sin\alpha'$. These latter resolved velocities will not be affected by the collision. The former will be affected exactly as if the sphere O moving along COO' with a velocity $a\cos\alpha$ were to impinge directly upon the sphere O' moving with a smaller velocity $a'\cos\alpha'$ estimated in the same direction. Hence if v, v', denote the resolved parts of the velocities after collision parallel to the line COO', we have, as may be readily ascertained by the principles of this chapter,

$$v = \frac{ma\cos\alpha - m'a\cos\alpha}{m+m'} + \frac{2m'a'\cos\alpha'}{m+m'},$$

$$v' = \frac{m'a'\cos\alpha' - ma'\cos\alpha'}{m+m'} + \frac{2ma\cos\alpha}{m+m'}.$$

Let V, V', denote the velocity of the spheres O, O', after collision, and ϕ, ϕ' the angles which the directions of their motions make with OO'; then

$$V^2 = v^2 + a^2\sin^2\alpha, \quad V'^2 = v'^2 + a'^2\sin^2\alpha',$$

$$\tan\phi = \frac{a\sin\alpha}{v}, \quad \tan\phi' = \frac{a'\sin\alpha'}{v'},$$

their motions still taking place in the planes BOO', $B'O'O$.

Keill; *Introductio ad Veram Physicam*, Lect. 14.

(6) Two imperfectly elastic bodies are moving in the same direction along the same straight line with given velocities; the one overtakes the other and collision ensues; to find the velocities of the two bodies after collision.

If m, m', be the masses of the two bodies, e their common elasticity; a, a', their velocities before, and v, v', their velocities after collision,

$$v = \frac{ma + m'a'}{m + m'} - \frac{em'(a - a')}{m + m'},$$

$$v' = \frac{ma + m'a'}{m + m'} + \frac{em(a - a')}{m + m'}.$$

Maclaurin; *Choc des Corps*, p. 30, *Prix de l'Academie*, Tom. I.

(7) To find with what velocity a ball must impinge upon another equal ball moving with a given velocity, that the impinging ball may be reduced to rest by the collision, the common elasticity of the balls being known.

If e be the common elasticity, and a the velocity of the ball which is struck, the impinging ball must impinge with an opposite velocity equal to

$$\frac{1 + e}{1 - e} a.$$

(8) To find the elasticity of two spheres A and B, and the ratio between their masses, that, when A impinges upon B at rest, A may be reduced to rest, and B move on with the n^{th} part of A's velocity.

If m, m', denote the masses of A, B, and e their common elasticity, then

$$e = \frac{1}{n}, \quad \frac{m'}{m} = n.$$

(9) Two perfectly elastic spheres meet directly with equal velocities; to find the relation between their magnitudes, that after collision one of them may remain at rest.

If m, m', denote their masses, m' corresponding to the one which remains at rest,

$$m' : m :: 3 : 1.$$

(10) To determine the velocities of two bodies A and B of given elasticity and given masses moving in the same direction, that after collision A may remain at rest and B may move along with an assigned velocity.

If m, m', be the masses of A, B, e their elasticity, β the velocity which B is to have after collision, and a, b, the required velocities of A, B, before impact,

$$a = \frac{1+e}{e}\frac{m'\beta}{m+m'}, \quad b = \frac{(em'-m)\beta}{e(m+m')}.$$

Maclaurin; *Choc des Corps*, p. 52, *Prix de l'Acad.* Tom. I.

(11) A spherical body A impinges directly with a certain velocity upon a spherical body B at rest, the common elasticity of the two bodies being given; to find the mass of a third body which, moving with the velocity which A has before the collision, shall have the same momentum which B has after the collision.

If m, m', denote the masses of A, B; e the common elasticity of A, B; and m'' the mass of the required body,

$$m'' = (1+e)\frac{mm'}{m+m'}.$$

(12) A, B, C, are three perfectly elastic balls in the same straight line, the masses of which are as 2, 3, 5: B and C being at rest, A impinges on B with a velocity 1, and B is thus made to impinge on C: to find the velocity of B, after the first impact, and of B and C after the second impact; and to ascertain whether A and B ever come together again.

After the first impact, B's velocity is $\frac{4}{5}$, and, after the second impact, C's velocity is $\frac{3}{5}$, and B's, in an opposite direction, is $\frac{1}{5}$.

A and B part to meet no more.

(13) A number of balls of given elasticity A_1, A_2, A_3, ... are placed in a line; A_1 is projected with a given velocity so as to impinge on A_2; A_2 then impinges on A_3, and so on; to find the masses of the balls A_2, A_3, ... in order that each of the balls A_1, A_2, A_3, ... may be at rest after impinging on the next; and to find the velocity of the n^{th} ball after its collision with the $(n-1)^{\text{th}}$.

If u_1 = the original velocity of A_1, the final velocity of the n^{th} ball is equal to $e^{n-1}u_1$; also

$$A_r = \frac{A_1}{e^{r-1}}.$$

(14) Any number of spheres of given elasticity being arranged in a straight line, and one of the extreme ones having a given velocity communicated to it so as to bring it into direct collision with the adjacent sphere of the series; to determine the velocity ultimately acquired by the last sphere.

If r be the number of the spheres, e their common elasticity, m_1, m_2, m_3, ... m_r, their masses, and a the velocity with which the first is projected; then, v being the velocity acquired at last by m_r,

$$v = (1+e)^{r-1} \frac{m_1}{m_1+m_2} \cdot \frac{m_2}{m_2+m_3} \cdot \frac{m_3}{m_3+m_4} \cdots\cdots\cdots \frac{m_{r-1}}{m_{r-1}+m_r} \cdot a.$$

Maclaurin; *Choc des Corps*, p. 54, *Prix de l'Acad.* Tom. I.

(15) A number of equal spheres are placed on a smooth table in a straight line and close together; they are connected together by equal inelastic threads; a motion is given to the first in the direction of the line which they form so as to separate it from the second; to find the time which elapses before the last sphere is put in motion.

If n be the number of spheres, a the length of each of the connecting threads, and β the velocity with which the first sphere is projected, then

$$\text{the time required} = \frac{n(n-1)}{1.2}\frac{a}{\beta}.$$

(16) To find the sum of the *vires vivæ* of two perfectly elastic bodies after direct collision.

If a, a', be the velocities before, and v, v', after collision,

$$mv^2 + m'v'^2 = ma^2 + m'a'^2.$$

Huyghens; *De Motu Corporum ex Percuss.* Prop. XI.
John Bernoulli; *Discours sur le Mouvement*, chap. X.

(17) To find the sum of the *vires vivæ* of two imperfectly elastic bodies after direct collision.

The notation remaining the same as in the preceding example, and e denoting the elasticity,

$$mv^2 + m'v'^2 = ma^2 + m'a'^2 - \frac{(1-e^2)\,mm'\,(a-a')^2}{m+m'},$$

which shews that *vis viva* is lost by the collision.

(18) Two balls, of elasticity e, are projected along a smooth fixed tube in the form of any closed curve, lying in a horizontal plane, from any two points in the tube: supposing u, v, to be the velocities of projection, estimated in the same direction, and c to be the length of the tube, to find the whole interval of time between the 1st and $(n+1)^{\text{th}}$ collisions.

The required interval is equal to

$$\frac{c}{u-v}\cdot\frac{e^{-n}-1}{1-e}.$$

(19) Three equal balls A, B, C, of elasticity e, are placed in order on a smooth horizontal plane in a straight line: velocities are impressed upon them in the direction ABC, those of A and C being each greater than that of B: two collisions having taken place, the velocities of A and B are observed to be equal to each other: to determine the ratio of the initial relative velocity of C, B, to that of A, B.

The required ratio is equal to

$$\frac{1}{2}\,\frac{(1-e)^2}{1+e}.$$

(20) Between two spheres of given masses is placed a row of spheres; a velocity is communicated to one of the original spheres so as to bring it into direct collision with the nearest of the intermediate ones; to find the requisite magnitudes of the intermediate spheres in order that the velocity acquired by the

last sphere may be the greatest possible, and to determine this velocity when their number becomes indefinitely great.

The intermediate spheres must be geometrical means between the two original ones; and if m, m', denote the masses of the original spheres, a the velocity communicated to m, and a' that acquired by m' when the number of the intermediate spheres becomes infinite,

$$a' = \left(\frac{m}{m'}\right)^{\frac{1}{2}} a.$$

(21) An imperfectly elastic sphere impinges upon a plane; to find the angles of incidence and of reflection, that the velocity before may be to the velocity after impact as $2^{\frac{1}{2}} : 1$, the elasticity being equal to $\frac{1}{3^{\frac{1}{2}}}$.

The angle of incidence $= \frac{1}{6}\pi$, the angle of reflection $= \frac{1}{4}\pi$.

(22) The edge of a smooth table is environed by a vertical border: supposing a perfectly elastic ball to be projected along the table from one of its foci, in a direction inclined at a given angle to the major axis, to find the inclination of its path to the same axis between the n^{th} and $(n+1)^{\text{th}}$ impacts.

If θ, θ_n, represent the given and required angles respectively, and e the eccentricity of the ellipse,

$$\tan\frac{\theta_n}{2} = \left(\frac{1-e}{1+e}\right)^n . \tan\frac{\theta}{2}.$$

(23) An inelastic sphere A, moving with a given velocity, impinges upon an inelastic sphere B at rest, the line joining the centres of the two spheres at the instant of collision making a given angle with the direction of A's motion; to determine the velocity of A after collision.

If α be the given angle; m, m', the masses of the spheres A, B; and a, v, the velocities of A before and after collision,

$$v = a\left\{\sin^2\alpha + \frac{m^2}{(m+m')^2}\cos^2\alpha\right\}^{\frac{1}{2}}.$$

(24) A sphere A in motion is struck by an equal one B moving with the same velocity, and in a direction making an angle α with that in which A is moving, in such a manner that the line joining their centres at the time of impact is in the direction of B's motion; to find the velocities of the spheres after impact, and to determine for what value of α that of A will be a maximum, the common elasticity of the spheres being supposed to be known.

If a denote the velocity of each of the spheres A, B, before, and u, v, their respective velocities after impact, then, e being their common elasticity,

$$u^2 = \tfrac{1}{4}a^2 \{1 + e + (1 - e) \cos \alpha\}^2 + a^2 \sin^2 \alpha,$$

$$v^2 = \tfrac{1}{4}a^2 \{1 - e + (1 + e) \cos \alpha\}^2.$$

When u is a maximum, $\qquad \cos \alpha = \dfrac{1 - e}{3 - e}.$

(25) Three perfectly elastic spheres A, B, C, are placed at the three angles of a plane triangle of which the angles are known; to compare the magnitudes of the spheres, when A impinging obliquely upon B is reflected so as to strike C, and thence reflected to its first position; the lines joining the centres of the spheres A, B, and A, C, during collision, being respectively perpendicular to the opposite sides of the triangle, and their diameters being inconsiderable in comparison with the sides of the triangle.

If m, m', m'', denote the masses of the spheres A, B, C; and α, β, γ, the angles of the triangle at which A, B, C, are placed,

$$\frac{m'}{m} = \frac{\sin \beta}{\sin (\alpha - \gamma)}, \quad \frac{m''}{m} = \frac{\sin \gamma}{\sin (\beta - \alpha)}.$$

(26) A sphere A (fig. 116), moving in the direction EAF with an assigned velocity impinges upon a sphere B at rest, the two spheres having the same elasticity; supposing AF' to be the direction of A's motion after impact, and $KABL$ to be a

straight line passing through the centres of the two spheres at the time of collision, to find the value of the angles *EAK* and *FAF′* when the latter angle has its greatest value.

If n be the ratio of the mass of A to that of B, e be the common elasticity of the spheres, $\angle EAK = \theta$, $\angle FAF' = \phi$; then

$$\tan\theta = \left(\frac{n-e}{n+1}\right)^{\frac{1}{2}}, \quad \tan\phi = \frac{\frac{1}{2}(1+e)}{\{(n+1)(n-e)\}^{\frac{1}{2}}}.$$

CHAPTER II.

RECTILINEAR MOTION OF A PARTICLE.

THE determination of the circumstances of the motion of a material particle, which moves in a straight line under the action of a finite accelerating or retarding force, depends upon the two following differential equations, called the equations of motion of the particle

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = f,$$

where t denotes the time of the motion reckoned from an assigned epoch, x the distance of the particle at the end of this time from an assigned point in the line of its motion, v the velocity, and f the accelerating or retarding force.

From these two equations we readily deduce the two following,

$$\frac{d^2x}{dt^2} = f, \quad v\frac{dv}{dx} = f.$$

These equations, which constitute the complete expression of the circumstances of rectilinear motion in the language of the differential calculus for every condition of acceleration or retardation, are due to Varignon, and were published in the *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 22. It may be observed however that, long before this, geometrical investigations of rectilinear motion for variable forces had been given by Newton[1].

From the formula $vdv = fdx$ we see that dv^2 varies as fdx: an opinion however was expressed by Daniel Bernoulli[2], that there is no reason to consider this the only possible law of variation; for instance, that we might as well have $dv^n \propto fdx$, n being any

[1] *Principia*, Lib. I. sect. 7; Lib. II. sect. 1.
[2] *Comment. Petrop.* 1727, p. 136.

quantity whatever. In opposition to Bernoulli's suggestion, Euler[1] endeavoured to prove that the law of the square of the velocity is necessarily true; and D'Alembert[2] shewed the truth of this law to depend simply upon the definition of the meaning of the symbol f.

The complete solution of a problem in rectilinear motion consists in the determination of relations between every two of the quantities x, v, f, t: now the general equations of rectilinear motion furnish us with only two independent relations between these four quantities; it is evident then that the data in every problem must consist in the expression of some particular equation, $\phi(x, v, f, t) = 0$ between x, v, f, t, so that we may have, in all, three equations connecting the four variables.

The function $\phi(x, v, f, t)$ may involve two, three, or all of the quantities x, v, f, t; and by the theory of combinations it is evident that there will be six varieties of the first, and four of the second class; hence the general problem of rectilinear motion resolves itself into eleven distinct classes of problems. We shall however confine ourselves to the consideration of those two classes in which the given function involves either x, f, alone; or x, f, v, alone: under the former head we shall exemplify the motion of a particle in vacuum; under the latter, in a resisting medium. The other classes are devoid of any physical interest.

SECT. 1. *Motion in Vacuum.*

(1) A particle is placed at a centre of repulsive force which varies as any power of the distance; to determine its velocity after receding to any distance from the centre, and the time of the motion.

Let μ represent the absolute force, x the distance of the particle from the centre of force after a time t, and v the velocity. Then, for the motion, we have

$$v\frac{dv}{dx} = \mu x^n.$$

[1] *Mechanica*, Tom. I. p. 62 et seq.

[2] *Traité de Dynamique.*

Integrating with respect to x, and bearing in mind that $v = 0$ when $x = 0$, we have

$$\tfrac{1}{2} v^2 = \mu \int_0^x x^n \, dx = \frac{\mu}{n+1} x^{n+1},$$

and therefore $$v^2 = \frac{2\mu}{n+1} x^{n+1},$$

which gives the velocity for any value of x.

Again, $$\frac{dx}{dt} = v = \left(\frac{2\mu}{1+n}\right)^{\frac{1}{2}} x^{\frac{1}{2}(n+1)}:$$

hence, t being equal to zero when $x = 0$, there is

$$t = \left(\frac{1+n}{2\mu}\right)^{\frac{1}{2}} \int_0^x x^{-\frac{1}{2}(n+1)} \, dx$$
$$= \frac{2}{1-n} \left(\frac{1+n}{2\mu}\right)^{\frac{1}{2}} x^{\frac{1}{2}(1-n)}.$$

Euler; *Mechanica*, Tom. I. p. 123.

(2) A particle being attracted by a force varying inversely as the n^{th} power of the distance, to find the value of n when the velocity acquired from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired from a to $\frac{1}{4}a$.

Let μ denote the absolute force, x the distance of the particle from the centre of force after a time t, and v_1, v_2, the two velocities. Then, for the motion of the particle,

$$v \frac{dv}{dx} = -\frac{\mu}{x^n}.$$

Hence, for the former motion, v being equal to zero when $x = \infty$,

$$v_1^2 = -2\mu \int_\infty^a \frac{1}{x^n} \, dx = \frac{2\mu}{(n-1)\, a^{n-1}},$$

and, for the latter motion, since $v = 0$ when $x = a$,

$$v_2^2 = -2\mu \int_a^{\frac{1}{4}a} \frac{1}{a^n} \, dx = \frac{2\mu}{n-1} \left(\frac{4^{n-1}}{a^{n-1}} - \frac{1}{a^{n-1}}\right).$$

But by hypothesis v_1^2 is equal to v_2^2; hence

$$\frac{2\mu}{n-1}\,\frac{1}{a^{n-1}} = \frac{2\mu}{n-1}\,\frac{1}{a^{n-1}}\,(4^{n-1}-1),$$

and therefore $\quad 1 = 4^{n-1} - 1, \quad 4^{n-1} = 2;$

hence $\quad n = \frac{3}{2}.$

(3) Four equal attractive forces are placed in the corners of a square, their intensity varying as any function of the distance; a particle is placed in one of the diagonals of the square very near to its centre; to find the time of an oscillation.

Let O be the centre of the square (fig. 117), E the position of the particle after any time t from the commencement of the motion; let $OD = a$, $OE = x$, $AE = CE = r$. Then, for the motion of the particle, taking the sum of the forces acting upon it in the line OD, we have

$$\frac{d^2x}{dt^2} = -2\phi(r)\frac{x}{r} + \phi(a-x) - \phi(a+x),$$

and therefore, neglecting powers of the small quantity x higher than the first, we get, by Taylor's theorem,

$$\frac{d^2x}{dt^2} = -2\left\{\frac{\phi(a)}{a} + \phi'(a)\right\}x,$$

or, putting the coefficient of x equal to $-k$,

$$\frac{d^2x}{dt^2} = -kx.$$

The integral of this equation is evidently

$$x = C\cos(k^{\frac{1}{2}}t + \epsilon),$$

C and ϵ being constants: let β be the initial value of x:

then $\quad \beta = C\cos\epsilon:$

but $\dfrac{dx}{dt} = 0$ initially, and therefore $0 = C\sin\epsilon$:

hence $\quad x = \beta\cos(k^{\frac{1}{2}}t).$

Now as soon as $k^{\frac{1}{2}}t$ becomes equal to π, x becomes equal to $-\beta$, its greatest negative value. Hence, the time of a complete oscillation being T, we have

$$k^{\frac{1}{2}}T = \pi,$$

and therefore, substituting for k its value,

$$T = \frac{\pi}{\sqrt{2}} \left\{ \frac{1}{a} \phi(a) + \phi'(a) \right\}^{-\frac{1}{2}}.$$

(4) A particle A attracts a particle B with a force always to that with which B attracts A in the ratio of μ' to μ; the particles being originally at rest, to find their position as well as that of their centre of gravity after any time; the intensity of each force being directly as the distance between the particles.

Let O be a fixed point in the line of the motion of the particles, (fig. 118), and let $OA = x$, $OB = x'$, at any time t.

Then, for the motion, we have

$$\frac{d^2x}{dt^2} = \mu(x' - x) \qquad (1),$$

$$\frac{d^2x'}{dt^2} = -\mu'(x' - x) \qquad (2).$$

Multiplying (1) and (2) by μ' and μ respectively, and adding the results, we get

$$\mu' \frac{d^2x}{dt^2} + \mu \frac{d^2x'}{dt^2} = 0.$$

Integrating, and bearing in mind that $\frac{dx}{dt}, \frac{dx'}{dt}$, are both equal to zero initially,

$$\mu' \frac{dx}{dt} + \mu \frac{dx'}{dt} = 0;$$

integrating again,

$$\mu'x + \mu x' = \mu'a + \mu a' \qquad (3),$$

a, a', being the initial values of x, x'.

Again, subtracting (1) from (2),

$$\frac{d^2}{dt^2}(x' - x) + (\mu + \mu')(x' - x) = 0;$$

the integral of this equation is

$$x' - x = C \cos\{(\mu + \mu')^{\frac{1}{2}} t + \epsilon\},$$

C and ϵ being constants.

Now, initially, $x = a$, $x' = a'$, $\frac{dx}{dt} = 0$, $\frac{dx'}{dt} = 0$;

hence $\qquad a' - a = C \cos \epsilon, \qquad 0 = C \sin \epsilon:$

the integral therefore becomes

$$x' - x = (a' - a) \cos \{(\mu + \mu')^{\frac{1}{2}} t\} \quad\text{.............. (4).}$$

From (3) and (4) we readily obtain

$$x = \frac{\mu' a + \mu a'}{\mu' + \mu} - \frac{\mu (a' - a)}{\mu' + \mu} \cos \{(\mu' + \mu)^{\frac{1}{2}} t\},$$

$$x' = \frac{\mu' a + \mu a'}{\mu' + \mu} + \frac{\mu' (a' - a)}{\mu' + \mu} \cos \{(\mu' + \mu)^{\frac{1}{2}} t\};$$

$$(m + m') \bar{x} = mx + m'x'$$

$$= \frac{m' + m}{\mu' + \mu} (\mu' a + \mu a') + \frac{\mu' m' - \mu m}{\mu' + \mu} (a' - a) \cos \{(\mu' + \mu)^{\frac{1}{2}} t\},$$

where m, m', denote the masses of A, B, and $\bar{x}$ the distance of their centre of gravity from O at any time t.

If μ', μ, be proportional to m, m', respectively, then clearly from our general result

$$(m + m') \bar{x} = \frac{m' + m}{\mu' + \mu} (\mu' a + \mu a')$$

or

$$\bar{x} = \frac{\mu' a + \mu a'}{\mu' + \mu},$$

which shews that the centre of gravity remains stationary during the whole motion.

(5) A body not affected by gravity falls down the axis of a thin cylindrical tube infinite in length, the particles of which attract with a force which varies inversely as the square of the distance; to find the velocity acquired in falling through a given space.

Let k be the thickness of the tube, r the radius of its interior surface, x the distance of the particle P from the extremity of the tube after a time t; then the volume of a portion of the tube contained between slices at distances s and $s + ds$ from P will be $2\pi rk ds$, and therefore the attraction of this elemental portion on the particle along the axis of the tube will be, the unit of attraction being chosen to be the attraction of a unit of mass at a unit of distance,

$$2\pi\rho rkds \cdot \frac{1}{s^2+r^2} \cdot \frac{s}{(s^2+r^2)^{\frac{1}{2}}},$$

where ρ denotes the density; and therefore for the motion of the particle we have

$$\frac{d^2x}{dt^2} = 2\pi\rho rk \int_{-x}^{+\infty} \frac{sds}{(s^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi\rho rk \cdot {}_{-x}^{+\infty}\left\{-\frac{1}{(s^2+r^2)^{\frac{1}{2}}}\right\} = \frac{2\pi\rho rk}{(x^2+r^2)^{\frac{1}{2}}}:$$

multiplying by $2\frac{dx}{dt}$ and integrating,

$$v^2 = \frac{dx^2}{dt^2} = 4\pi\rho rk \log\{x + (x^2+r^2)^{\frac{1}{2}}\} + C.$$

But $v = 0$ when $x = 0$; hence

$$0 = 4\pi\rho rk \log r + C,$$

and therefore $$v^2 = 4\pi\rho rk \log \frac{x + (x^2+r^2)^{\frac{1}{2}}}{r}.$$

(6) Two balls are moving in a straight line, one of them only being acted on by a force; if the force be constant and tend always towards the other ball, to compare the times which elapse between consecutive impacts.

Let $v =$ the relative velocity of the balls just before the first impact: then $ev =$ their relative velocity just after: hence, in a time equal to $\frac{2ev}{f}$, the balls are again in contact, f denoting the force. Similarly, the next interval is $\frac{2e^2v}{f}$, the next $\frac{2e^3v}{f}$, and so on. Thus e is the ratio of the times between consecutive impacts.

(7) From a point A in a vertical line AB falls a particle from rest; at the same instant another particle is projected upwards from B with a given velocity: to find when and where the two particles will meet; the motion being supposed to take place in a vacuum, and gravity being the only force to which the particles are subject.

Let a be the length of the line AB, β the velocity of projection of the ascending particle, x the distance from A at which collision takes place, and t the time of this event from the commencement of the motion. Then

$$x = \frac{ga^2}{2\beta^2}, \quad t = \frac{a}{\beta}.$$

Kurdwanowski; *Mém. de l'Acad. des Sciences de Berlin*, 1755, p. 394.

(8) A body is projected vertically upwards with a velocity $4g$: after two seconds, gravity ceases to act for one second, and is then doubled: to find the greatest height to which the body ascends, and to determine the velocity when it returns to the point of projection.

The greatest height and the required velocity are respectively $9g$ and $6g$.

(9) A body of known elasticity falls from a given altitude above a hard horizontal plane, and rebounds continually till its whole velocity is destroyed; to find the whole space described.

If a denote the first altitude, e the elasticity, and s the required space,

$$s = \frac{1+e^2}{1-e^2}a.$$

(10) Two perfectly elastic balls beginning to descend from different points in the same vertical line impinge upon a perfectly hard plane inclined at an angle of 45°, and move along a horizontal plane with the velocities acquired; to find what distance they will move along the horizontal plane before collision takes place.

If a, a', denote the altitudes through which they fall, and s the distance required,

$$s = 2\,(aa')^{\frac{1}{2}}.$$

(11) A particle falling in a straight line towards a centre of force, the intensity of which varies as the n^{th} power of the distance, acquires a velocity β on arriving at a distance a from the centre; to find at what distance z from the centre of force it must have commenced its motion.

Let μ denote the absolute force; then z will be given by the equation

$$z^{n+1} - a^{n+1} = \frac{n+1}{2\mu}\beta^2.$$

Euler; *Mechan.* Tom. I. p. 109.

(12) A particle falls towards a centre of force, of which the intensity varies inversely as the cube of the distance; to find the whole time of descent.

Let μ denote the absolute force and a the initial distance; then

$$\text{the time of descent} = \frac{a^2}{\mu^{\frac{1}{2}}}.$$

(13) A particle descends from an infinite distance towards a centre of force which varies inversely as the square of the distance; to find the velocity at a given distance from the centre of force.

Let μ be the absolute force and a the given distance; then

$$\text{the velocity} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}}.$$

(14) A body is projected vertically from the surface of the Earth; to find the height to which it will ascend.

If $g =$ the force of gravity at the Earth's surface, $r =$ the Earth's radius, and $V =$ the velocity of the body's projection, then the height of ascent is equal to

$$\frac{2gr^2}{2gr - V^2}.$$

If $V > (2gr)^{\frac{1}{2}}$, the body will never descend.

(15) A body falls from a given point towards a centre of force, the attraction at any distance r being $\frac{\mu}{r^{\frac{3}{2}}}$: to find the whole time of descent.

$$\text{The required time} = \frac{3\pi a^{\frac{5}{4}}}{8\mu^{\frac{1}{2}}}.$$

(16) A body moves from rest at a distance a towards a centre of force, the force varying inversely as the distance: to determine the value of β in order that the time of describing the space between βa and $\beta^n a$ may be a maximum.

The required value of β is equal to $n^{-\frac{1}{2(n-1)}}$.

(17) A particle is placed at an assigned point between two centres of force of equal intensity attracting directly as the distance; to determine the position of the particle at any time, and the period of its oscillations.

Let a denote the initial distance of the particle from the middle point of the line joining the two centres of force, x the distance after the expiration of a time t, and μ the absolute force of each centre. Then

$$x = a \cos \{(2\mu)^{\frac{1}{2}} t\}, \text{ and the period of an oscillation} = \frac{\pi}{(2\mu)^{\frac{1}{2}}}.$$

(18) A particle acted upon by two central forces, each attracting with an intensity varying inversely as the square of the distance, is projected from an assigned point between them towards one of the centres; to find the velocity of projection that the particle may just arrive at the neutral point of attraction and remain at rest there.

Let μ^2, μ'^2, denote the absolute forces of the two centres; $2a$, $2a'$, the initial distances of the particle from the two centres; and V the velocity of projection. Then

$$V = \frac{\dfrac{\mu}{a} - \dfrac{\mu'}{a'}}{\left(\dfrac{1}{a} + \dfrac{1}{a'}\right)^{\frac{1}{2}}}.$$

(19) A centre of force C (fig. 119) moves along the straight line OA with a uniform velocity, attracting, with a force varying directly as the first power of the distance, a particle P which is moving in the same straight line; having given the initial position of C, and both the initial position and the initial velocity of P, to find the position of P at any time.

Let a, a', be the initial distances of C, P, from O; β the uniform velocity of C, and β' the initial velocity of P; x the distance of P from O after a time t; μ the absolute force of attraction.

Then $$x = a + \beta t + \frac{\beta' - \beta}{\mu^{\frac{1}{2}}} \sin (\mu^{\frac{1}{2}} t) + (a' - a) \cos (\mu^{\frac{1}{2}} t).$$

Riccati; *Bonon. Institut.* Tom. VI. p. 138; 1783.

(20) The circumstances remaining the same as in the preceding problem, except that the force is repulsive; to find the position of P at any time.

$$x = a + \beta t - \{\mu^{\frac{1}{2}} (a - a') + (\beta - \beta')\} \frac{e^{\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}} - \{\mu^{\frac{1}{2}} (a - a') - (\beta - \beta')\} \frac{e^{-\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}}.$$

Riccati; *Ib.* p. 151.

(21) Supposing the centre C to move along OA with a uniform acceleration, attracting directly as the distance; to determine the place of P at any time.

Let f represent the increment of C's velocity in each unit of time, and β its velocity at the commencement of the motion; then, the notation remaining the same as in the two preceding problems,

$$x = a - \frac{f}{\mu} + \beta t + \tfrac{1}{2} f t^2 + \frac{\beta' - \beta}{\mu^{\frac{1}{2}}} \sin (\mu^{\frac{1}{2}} t) + \left(a' - a + \frac{f}{\mu}\right) \cos (\mu^{\frac{1}{2}} t).$$

Riccati; *Ib.* p. 168.

(22) The circumstances and notation remaining the same as in the preceding example, except that the force is repulsive; to find the place of P at any time.

$$\begin{aligned} x = a + \frac{f}{\mu} + \beta t + \tfrac{1}{2} f t^2 & \\ & - \left\{\mu^{\frac{1}{2}} \left(a - a' + \frac{f}{\mu}\right) + (\beta - \beta')\right\} \frac{e^{\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}} \\ & - \left\{\mu^{\frac{1}{2}} \left(a - a' + \frac{f}{\mu}\right) - (\beta - \beta')\right\} \frac{e^{-\mu^{\frac{1}{2}} t}}{2\mu^{\frac{1}{2}}}. \end{aligned}$$

Riccati; *Ib.* p. 182.

(23) A particle is placed at a given distance from a uniform thin plate of indefinite extent, every particle of which attracts with a force varying inversely as the square of the distance; to find the time in which the particle will arrive at the surface of the plate.

Let k denote the thickness of the plate, ρ its density, and a the initial distance of the particle from it: then

$$\text{the time} = \left(\frac{a}{\pi\rho k}\right)^{\frac{1}{2}}.$$

(24) A particle being placed at a given distance from a thin circular lamina of uniform density, in a line passing through its centre and perpendicular to its plane, to find the velocity which it will acquire by moving to the lamina, the attractive force of each molecule of the lamina varying inversely as the square of the distance.

Let a be the radius of the circular lamina, k its thickness, ρ its density, b the given distance; then, the unit of attraction being the attraction of a unit of mass at a unit of distance, and V being the velocity required,

$$V^2 = 4\pi\rho k\,\{a + b - (a^2 + b^2)^{\frac{1}{2}}\}.$$

(25) A particle is placed at a small distance from the centre of a thin ring of uniform density and thickness, every molecule of which repels with a force varying inversely as the square of the distance; to determine the position of the particle at any time, and the period of its oscillations.

Let l be the initial distance of the particle from the centre of the ring, a the radius of the ring, k the area of a section, ρ the density, and x the distance of the particle from the centre at the end of a time t. Then, the repulsion of a unit of the ring's mass at a unit of distance being taken as the unit of repulsion,

$$x = l\cos\left\{\frac{(\pi\rho k)^{\frac{1}{2}}}{a}\,t\right\} \text{ and the period of an oscillation} = \left(\frac{\pi}{\rho k}\right)^{\frac{1}{2}} a.$$

SECT. 2. *Motion in Resisting Media.*

The retardation experienced by a material particle in traversing a resisting medium of variable density, depends at any point of its path upon the density of the medium and the velocity of the particle, and will therefore be some function of these quantities. The nature of this function can be ascertained only by experiment. In mathematical investigations, for the sake of simplicity and as a probable approximation to the truth, the function is assumed to be of the form $k\rho\Omega$, where ρ denotes the density of the medium and Ω some function of the velocity of the particle; and where k is an invariable coefficient depending upon the nature of the particular medium in respect to the tenacity and the friction of its constituent molecules.

For the earliest mathematical development of the theory of the resistance of media to the motion of bodies, we are indebted to the labours of Newton and Wallis. The profound researches of Newton on this theory were published in the year 1687 in the second book of the *Principia*. In the same year, after the publication of Newton's investigations, Wallis, who had independently arrived at valuable conclusions on the subject, communicated his reflections to the Royal Society, which were inserted in the *Philosophical Transactions* for the year 1687. There is a paper by Leibnitz on the question of resisting media in the *Acta Erudit. Lips.* ann. 1689, in which he developes opinions which he declares to have been communicated by him twelve years before to the Royal Academy of Sciences. Huyghens also has discussed certain points of the theory at the end of his *Discours de la Cause de la Pesanteur*, published in the year 1690. Finally, all which these philosophers had communicated to the scientific world either with or without demonstration, was investigated analytically by Varignon in a series of papers in the *Mémoires de l'Acad. des Sciences de Paris*, for the years 1707, 1708, 1709, and 1710. There is an elaborate paper by Bouguer in the *Mém. de l'Acad. des Sciences de Paris*, 1731, p. 390, in which he investigates the motion of a particle in resisting media which are themselves in motion.

(1) A particle acted upon by no forces is projected with a given velocity in a resisting medium of uniform density, where the resistance varies directly as the velocity; to determine the velocity and the space described at the end of any time.

For the motion of the particle we have

$$v\frac{dv}{dx} = -\mu v,$$

where μ is some constant quantity; hence

$$\frac{dv}{dx} = -\mu,$$

$$v = C - \mu x.$$

Let β denote the initial velocity when x is supposed to be zero; then $\beta = C$, and therefore

$$v = \beta - \mu x;$$

whence, v being equal to $\frac{dx}{dt}$, we have

$$dt = \frac{dx}{\beta - \mu x},$$

$$t = C - \frac{1}{\mu}\log(\beta - \mu x).$$

But $t = 0$ when $x = 0$; and therefore

$$0 = C - \frac{1}{\mu}\log\beta;$$

hence
$$t = \frac{1}{\mu}\log\frac{\beta}{\beta - \mu x},$$

$$\epsilon^{\mu t} = \frac{\beta}{\beta - \mu x},$$

$$x = \frac{\beta}{\mu}(1 - \epsilon^{-\mu t}),$$

and therefore
$$v = \beta\epsilon^{-\mu t}.$$

Newton; *Principia*, Lib. II. Prop. 1 and 2. Leibnitz; *Acta Erudit. Lips.* ann. 1689. Varignon; *Mém. de l'Acad. des Sciences de Paris*, ann. 1707, p. 391.

(2) A body falls towards a centre of force which varies as the inverse cube of the distance, in a medium of which the density varies also as the inverse cube, and of which the resistance varies as the square of the velocity; to find the velocity at any distance from the centre.

Let x represent the distance of the particle from the centre after a time t, and let a be the initial distance. Let k denote the force of resistance at a unit of distance for a unit of velocity, and μ the absolute force of attraction.

Then, for the motion of the particle,

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} + \frac{k}{x^3}\frac{dx^2}{dt^2},$$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2\mu}{x^3}\frac{dx}{dt} + \frac{2k}{x^3}\frac{dx^3}{dt^3},$$

$$\frac{d}{dt}\frac{dx^2}{dt^2} = \mu\frac{d}{dt}\frac{1}{x^2} - k\frac{dx^2}{dt^2}\frac{d}{dt}\frac{1}{x^2}.$$

Assume $\dfrac{dx^2}{dt^2} = w$ and $\dfrac{1}{x^2} = z$; then

$$\frac{dw}{dt} = \mu\frac{dz}{dt} - kw\frac{dz}{dt},$$

$$dw + kwdz = \mu dz,$$

$$d(\epsilon^{kz}w) = \mu\epsilon^{kz}dz,$$

$$\epsilon^{kz}w = C + \frac{\mu}{k}\epsilon^{kz}.$$

Hence, putting for w and z their values,

$$\epsilon^{\frac{k}{x^2}}v^2 = C + \frac{\mu}{k}\epsilon^{\frac{k}{x^2}}.$$

But $x = a$ when $v = 0$; hence

$$0 = C + \frac{\mu}{k}\epsilon^{\frac{k}{a^2}},$$

$$\epsilon^{\frac{k}{x^2}}v^2 = \frac{\mu}{k}\left(\epsilon^{\frac{k}{x^2}} - \epsilon^{\frac{k}{a^2}}\right),$$

$$v^2 = \frac{\mu}{k}\left\{1 - \epsilon^{-k\left(\frac{1}{x^2}-\frac{1}{a^2}\right)}\right\}.$$

When x becomes equal to infinity, suppose V to be the value of v; then

$$V^2 = \frac{\mu}{k}(1 - \epsilon^{\frac{k}{a^2}}).$$

V is called the terminal velocity of the particle, a technicality invented by Huyghens[1], to signify the ultimate velocity of a particle descending in a resisting medium to an indefinitely great depth.

(3) To determine the motion of a particle, not acted upon by any force, when the resistance varies as any power of the velocity.

For the determination of the relation between the velocity and the space,

$$v\frac{dv}{dx} = -kv^n,$$

$$kdx = -\frac{dv}{v^{n-1}},$$

$$kx = C + \frac{1}{(n-2)\,v^{n-2}}.$$

Let $x = 0$ and $v = \beta$ initially; then

$$0 = C + \frac{1}{(n-2)\,\beta^{n-2}},$$

$$(n-2)\,kx = \frac{1}{v^{n-2}} - \frac{1}{\beta^{n-2}} \dots\dots\dots\dots\dots\dots (1).$$

Again, for the relation between the velocity and time,

$$\frac{dv}{dt} = -kv^n, \quad kdt = -\frac{dv}{v^n},$$

$$kt = C + \frac{1}{(n-1)\,v^{n-1}}.$$

But $v = \beta$ when $t = 0$; hence

$$0 = C + \frac{1}{(n-1)\,\beta^{n-1}},$$

$$(n-1)\,kt = \frac{1}{v^{n-1}} - \frac{1}{\beta^{n-1}} \dots\dots\dots\dots\dots\dots (2).$$

[1] *Discours sur la Cause de la Pesanteur*, p. 170.

If between (1) and (2) we eliminate v, we shall obtain a relation between s and t.

Varignon; *Mém. de l'Acad. des Sciences de Paris*, 1707, p. 404.

(4) A particle acted on by gravity falls from a given altitude in a medium of uniform density, where the resistance varies as the square of the velocity; on arriving at the lowest point of its descent it is reflected upwards with the velocity which it has acquired in its fall; after reaching its greatest altitude it again descends and is again reflected; and so on perpetually: to determine the altitude of ascent after any number of reflections.

Let the maximum altitudes of the particle be represented by $a_1, a_2, a_3, \ldots\ldots, a_1$ being the altitude from which it originally falls. Let c denote the volume of the particle, and ρ, ρ', the density of the particle and of the fluid.

For the descent down any of the altitudes there is

$$v\frac{dv}{dx} = \frac{cg\rho - cg\rho'}{c\rho} - kv^2,$$

or

$$v\frac{dv}{dx} = g' - kv^2, \quad \text{where } g' = \left(1 - \frac{\rho'}{\rho}\right)g,$$

$$\frac{vdv}{g' - kv^2} = dx,$$

$$-\frac{1}{2k}\log(g' - kv^2) = x + C;$$

but, the origin of x being the highest point,

$$-\frac{1}{2k}\log g' = C;$$

hence

$$\frac{1}{2k}\log\frac{g'}{g' - kv^2} = x,$$

and therefore, if v_n denote the velocity acquired down the n^{th} altitude,

$$\log\frac{1}{1 - \dfrac{k}{g'}v_n^2} = 2ka_n,$$

$$\frac{k}{g'}v_n^2 = 1 - e^{-2ka_n} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For the ascent up the $(n+1)^{\text{th}}$ altitude, the origin of x being the lowest point,

$$v\frac{dv}{dx} = -g' - kv^2, \qquad \frac{vdv}{g' + kv^2} = -dx,$$

$$\frac{1}{2k}\log(g' + kv^2) = C - x,$$

$$\frac{1}{2k}\log(g' + kv_n^2) = C,$$

$$\frac{1}{2k}\log g' = C - a_{n+1},$$

$$\frac{1}{2k}\log\left(1 + \frac{k}{g'}v_n^2\right) = a_{n+1},$$

$$\frac{k}{g'}v_n^2 = \epsilon^{2ka_{n+1}} - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, from (1) and (2),

$$\epsilon^{2ka_{n+1}} + \epsilon^{-2ka_n} = 2:$$

assume $\epsilon^{2ka_n} = u_n$, and we have

$$u_{n+1} + \frac{1}{u_n} = 2,$$

$$u_n u_{n+1} + 1 = 2u_n.$$

Putting $u_n = v_n + 1$, we get

$$(v_n + 1)(v_{n+1} + 1) + 1 = 2(v_n + 1),$$

$$v_n v_{n+1} + v_{n+1} - v_n = 0, \qquad \frac{1}{v_{n+1}} - \frac{1}{v_n} = 1,$$

$$\Delta\frac{1}{v_n} = 1, \qquad \frac{1}{v_n} = n + C,$$

$$\frac{1}{u_n - 1} = n + C, \qquad u_n - 1 = \frac{1}{n + C}, \qquad u_n = \frac{n + 1 + C}{n + C}.$$

But $\dfrac{1}{u_1 - 1} = 1 + C$; hence

$$u_n = \frac{n + \dfrac{1}{u_1 - 1}}{n - 1 + \dfrac{1}{u_1 - 1}} = \frac{nu_1 - n + 1}{(n-1)u_1 - n + 2}.$$

Or, putting for u_n, u_1, their values,

$$\epsilon^{2ka_n} = \frac{n\epsilon^{2ka_1} - n + 1}{(n-1)\,\epsilon^{2ka_1} - n + 2},$$

$$a_n = \frac{1}{2k} \log \frac{n\epsilon^{2ka_1} - n + 1}{(n-1)\,\epsilon^{2ka_1} - n + 2}.$$

If a_1 be equal to infinity,

$$a_n = \frac{1}{2k} \log \frac{n}{n-1}.$$

Euler; *Mechan.* Tom. I. p. 192.

(5) To determine the centripetal force that a particle may always descend to a given centre in the same time from whatever distance it commences its motion; the density of the medium in which the particle moves being known at every point in its path, and the resistance varying as the square of the velocity.

The equation of motion is

$$v\frac{dv}{dx} = -p + kv^2,$$

where p denotes the centripetal force, and k the density at any point. Multiplying by

$$2\epsilon^{-2\int k dx}\, dx,$$

the equation becomes

$$d\left(\epsilon^{-2\int k dx}\, v^2\right) = -\, 2\,\epsilon^{-2\int k dx}\, p dx.$$

Integrating, $$\epsilon^{-2\int k dx}\, v^2 = C - 2\int \epsilon^{-2\int k dx}\, p dx.$$

Let a denote the initial distance of the particle from the centre of force; then, the velocity being initially equal to zero,

$$\epsilon^{-2\int k dx}\, v^2 = A - X,$$

where $A = 2\int_0^a \epsilon^{-2\int k dx}\, p dx$ and $X = 2\int_0^x \epsilon^{-2\int k dx}\, p dx.$

Therefore $$v = (A - X)^{\frac{1}{2}}\, \epsilon^{\int k dx},$$

$$dt = -\frac{\epsilon^{-\int k dx}\, dx}{(A - X)^{\frac{1}{2}}}.$$

Now since X, k, are both functions of x, it is clear that we may assume

$$\epsilon^{-\int k dx} dx = \frac{dX}{P}$$

where P is a function of X alone; hence

$$dt = -\frac{dX}{P(A-X)^{\frac{1}{2}}},$$

and the whole time of descent, since $X=0$ when $x=0$, will be equal to

$$-\int_a^0 \frac{\epsilon^{-\int k dx} dx}{(A-X)^{\frac{1}{2}}} = -\int_A^0 \frac{dX}{P(A-X)^{\frac{1}{2}}};$$

and since the value of this integral is to be the same for all values of a and therefore of A, the differential

$$\frac{dX}{P(A-X)^{\frac{1}{2}}}$$

must be of no dimensions in X, dX, and A. Hence we must have P, which clearly cannot involve a, equal to $\frac{X^{\frac{1}{2}}}{\beta}$, where β is some constant quantity; and therefore

$$\beta \frac{dX}{X^{\frac{1}{2}}} = \epsilon^{-\int k dx} dx;$$

hence, X and x being simultaneously equal to zero,

$$2\beta X^{\frac{1}{2}} = \int_0^x \epsilon^{-\int k dx} dx, \quad 4\beta^2 X = \left\{\int_0^x \epsilon^{-\int k dx} dx\right\}^2,$$

$$8\beta^2 \int_0^x (\epsilon^{-2\int k dx} p\, dx) = \left\{\int_0^x \epsilon^{-\int k dx} dx\right\}^2,$$

$$4\beta^2 \epsilon^{-2\int k dx} p\, dx = \epsilon^{-\int k dx} dx \int_0^x \epsilon^{-\int k dx} dx,$$

$$p = \frac{1}{4\beta^2} \epsilon^{\int k dx} \int_0^x \epsilon^{-\int k dx} dx.$$

If k be equal to zero, which corresponds to a perfect vacuum,

$$p = \frac{1}{4\beta^2} \int_0^x dx = \frac{x}{4\beta^2},$$

or the centripetal force varies as the distance.

If the medium be uniform, or k a constant quantity,

$$p = \frac{1}{4\beta^2} \epsilon^{kx} \int_0^x \epsilon^{-kx}\, dx$$

$$= \frac{1}{4\beta^2} \epsilon^{kx} . \frac{1}{k} (1 - \epsilon^{-kx}) = \frac{1}{4\beta^2 k} (\epsilon^{kx} - 1).$$

Euler; *Mechan.* Tom. I. p. 220.

(6) A centre of force C, (fig. 120), moves along the straight line OA with a uniform velocity, repelling with a force varying directly as the first power of the distance; a particle P is moving along the same straight line OA in a medium resisting as the velocity; having given the initial position of C, and both the initial position and the initial velocity of P, to find the position of P at any time.

Let β be the uniform velocity of C, a its initial distance from O; x the distance of P from O at the end of the time t; μ the absolute force of repulsion; k the resistance of the medium for a unit of velocity. Then, the distance between C and P at the time t being $a + \beta t - x$, we have for the motion of P, so long as it continues to proceed in the direction OA,

$$\frac{dv}{dt} = -\mu (a + \beta t - x) - kv,$$

v being the velocity of P at the time t, estimated in the direction OA. Supposing the particle to be moving in the direction AO, then, v being also estimated in this direction, we should have

$$\frac{dv}{dt} = \mu (a + \beta t - x) - kv,$$

an equation deducible from the former one by the substitution of $-v$ in place of v: hence the former equation applies to the motion of the particle under all circumstances, the quantity v being supposed to involve the direction-sign of the velocity implicitly.

But $v = \frac{dx}{dt}$, and therefore

$$\frac{d^2x}{dt^2} = -\mu (a + \beta t - x) - k \frac{dx}{dt},$$

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = \mu(x - a - \beta t);$$

and therefore, putting $x - a - \frac{k\beta}{\mu} - \beta t = z$,

$$\frac{d^2z}{dt^2} + k\frac{dz}{dt} = \mu z.$$

Assume $z = Ae^{\rho t}$, A and ρ being constants; then, substituting for z in the differential equation, we have

$$\rho^2 + k\rho = \mu,$$

$$4\rho^2 + 4k\rho + k^2 = 4\mu + k^2,$$

$$2\rho = \quad -k \pm (4\mu + k^2)^{\frac{1}{2}};$$

hence the complete integral is

$$z = Ce^{\gamma t} + C'e^{\gamma' t},$$

where γ, γ', are the two values of ρ obtained by the solution of the quadratic. Hence for the position of P at any time we have

$$x = a + \frac{k\beta}{\mu} + \beta t + Ce^{\gamma t} + C'e^{\gamma' t}.$$

Suppose α', β', to be the initial values of x, $\frac{dx}{dt}$; then clearly

$$\alpha' = a + \frac{k\beta}{\mu} + C + C', \quad \beta' = \beta + \gamma C + \gamma' C';$$

from which two equations the values of the constants C and C' are immediately determined.

For further information on the subject of the rectilinear motion of a particle in a resisting medium under the action of a centre of force moving according to any assigned law, the reader is referred to Riccati; *De motu rectilineo corporis attracti aut repulsi a centro mobili; Disquisitio quarta. Comment. Bonon.* Tom. VI. p. 212; 1783.

(7) A particle is projected with a given velocity, towards a centre of force attracting inversely as the cube of the distance, in a medium of which the density varies inversely as the square of the distance from the centre of force; to determine

the velocity of the particle at any distance from the centre, the resistance for a given density varying as the square of the velocity.

If β denote the velocity of projection, μ the absolute attracting force; k the retarding force of the medium, at a unit of distance from the centre of force, for a unit of velocity; a the initial distance of the particle, and x its distance corresponding to a velocity v,

$$\epsilon^{-\frac{2k}{a}} v^2 - \epsilon^{-\frac{2k}{x}} \beta^2 = \frac{\mu}{2k^2}\left(\frac{a-2k}{a}\epsilon^{-\frac{2k}{x}} - \frac{x-2k}{x}\epsilon^{-\frac{2k}{a}}\right).$$

(8) A particle is projected with a given velocity in a uniform medium, in which the resistance varies as the square root of the velocity; to find what time will elapse before the particle is reduced to rest.

If β be the velocity of projection, and k the resistance for a unit of velocity,

$$\text{the required time} = \frac{2\beta^{\frac{1}{2}}}{k}.$$

(9) If t denote the time in which a particle falling from rest will acquire a certain velocity, and τ the time in which, when projected vertically upwards, it will lose the same velocity; the motion in both cases taking place in a medium of uniform density, where the resistance varies as the square of the velocity; to investigate the relation between t and τ.

If k denote the resistance for a unit of velocity,

$$\log \tan\left(\tfrac{1}{4}\pi + k^{\frac{1}{2}} g^{\frac{1}{2}} \tau\right) = 2g^{\frac{1}{2}} k^{\frac{1}{2}} t.$$

(10) A particle projected with a velocity of 1000 feet a second, loses half its velocity by passing through 3 inches of a resisting medium, in which the resistance is uniform; to find the time of passing through this space.

The required time = 3000th part of a second.

(11) A particle, attracted to a centre of constant attractive force, moves directly towards it from rest, through a medium of which the resistance varies as the square of the velocity directly,

and as the distance from the centre inversely; to find the velocity for any position of the particle during its approach towards the centre, and to ascertain its distance from the centre when its velocity is a maximum.

If f denote the constant central force, v the velocity for any distance x from the centre, a the initial value of x, k the resistance when x and v are each equal to unity, and x' the central distance when v is a maximum;

$$v^2 = \frac{2f}{1-2k} x^{2k} (a^{1-2k} - x^{1-2k}), \quad x' = (2k)^{\frac{1}{1-2k}} a.$$

(12) One particle begins to fall from the higher extremity of a vertical line, at the same instant in which another is projected upwards with a given velocity; the particles move in a uniform medium in which the resistance varies as the velocity; to find the time in which they will meet.

Let a denote the length of the vertical line, β the velocity with which the lower particle is projected upwards, k the resistance for a unit of velocity, and t the required time; then

$$t = \frac{1}{k} \log \frac{\beta}{\beta - ka}.$$

(13) A particle is projected vertically upwards in a medium in which the resistance is equal to kv^2; if V be the velocity of projection, to find the particle's velocity V_1, when it again arrives at the point of projection.

$$V_1^2 = \frac{gV^2}{g + kV^2}.$$

(14) A particle, of which the elasticity is e, falls from rest from an altitude a in a uniform medium, the resistance of which is kv^2; and impinging upon a perfectly hard horizontal plane, rises and falls alternately; to determine the whole space described before the motion ceases.

$$\text{The required space } = a + \frac{1}{k} \log \frac{1 - e^2 \epsilon^{-2ka}}{1 - e^2}.$$

Bordoni; *Memorie della Societa Italiana*, 1816, p. 162.

CHAPTER III.

FREE CURVILINEAR MOTION OF A PARTICLE.

SECT. 1. *Forces acting in any directions in one Plane.*

LET a particle moving in a plane curve under the action of any accelerating forces be referred to two fixed co-ordinate axes in the plane of its motion. Let x, y, be its co-ordinates at the end of a time t from an assigned epoch; and X, Y, the sum of the resolved parts of the accelerating forces parallel to the axes of x, y. Then the circumstances of the motion will be completely represented by the equations

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y \ldots\ldots\ldots\ldots (A.)$$

The method of resolving parallel to fixed axes the accelerating forces which act upon a particle, and thus reducing the determination of the circumstances of its motion to the formulæ for rectilinear acceleration, was first given by Maclaurin, *Treatise of Fluxions*, Vol. I. Art. 465, et sq., published in the year 1742. Before this time all problems in curvilinear motion were solved by the method of the tangential and normal resolutions, which, although more immediately suggested by the physical conception of the motion, is not generally so convenient in analysis as that of Maclaurin. The great work of Euler on Mechanics, which appeared in 1736, proceeds altogether by the ancient method of resolution. We shall devote the third section of this chapter to the illustration of the ancient equations of motion.

(1) A particle acted on by gravity is describing a path $KABL$, (fig. 121); having given the resolved part of the velocity at A at right angles to the chord AB, to find the resolved part at B taken in the same direction.

Let u be the given resolved part of the velocity at A; v the

velocity at right angles to AB at any point of the path corresponding to a time t from leaving A, and x the perpendicular distance of the particle from AB at this time; also let α be the inclination of AB to the horizon. Then, for the motion of the particle,

$$x = ut - \tfrac{1}{2}g \cos\alpha \,.\, t^2,$$

and
$$v = u - g \cos\alpha \,.\, t.$$

Now at the point B, $x = 0$, and therefore

$$u - \tfrac{1}{2}g \cos\alpha \,.\, t = 0, \quad g \cos\alpha \,.\, t = 2u;$$

hence for the value of v at B we have

$$v = -u.$$

Thus we see that the velocity of the particle at B, resolved at right angles to AB, is equal to the similarly resolved part at A, but of an opposite direction.

(2) A particle revolves in a parabola about a centre of force situated in the point of intersection of the directrix with the axis; to find the force at any point of the path of the particle.

Take the centre of force as the origin of co-ordinates, the axis of the parabola as the axis of x, and the directrix as the axis of y; let $4m$ be the principal parameter, r the distance of the particle at any time from the origin, and F the force estimated repulsively.

The equations of motion will be

$$\frac{d^2x}{dt^2} = \frac{Fx}{r}, \qquad \frac{d^2y}{dt^2} = \frac{Fy}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the parabola will be

$$y^2 = 4m(x - m) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

hence
$$y\frac{dy}{dt} = 2m\frac{dx}{dt} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$y\frac{d^2y}{dt^2} + \frac{dy^2}{dt^2} = 2m\frac{d^2x}{dt^2};$$

and therefore, from (1),

$$F(2mx - y^2) = r\frac{dy^2}{dt^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, eliminating F between the equations (1), we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

integrating, and adding a constant c which will represent twice the area described in a unit of time about the centre of force, we obtain

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c;$$

and therefore, by (3),

$$(2mx - y^2)\frac{dy}{dt} = 2mc;$$

hence, from (4),

$$F = \frac{4m^2c^2r}{(2mx - y^2)^3} = \frac{c^2r}{2m(2m - x)^3}, \text{ by (2).}$$

(3) A particle urged towards a plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity; to determine the force which must act at the same time on the particle parallel to the plane, that it may move in a given parabola having its axis in the plane, and to find the co-ordinates of the particle at any epoch of the motion in terms of the time.

Let the initial place of the particle be taken as the origin of co-ordinates, the axis of the parabola as the axis of x, and a straight line at right angles to the plane through the origin as the axis of y. Then, since the required force must evidently act parallel to the axis of x, we have, by Maclaurin's Equations (A),

$$\frac{d^2x}{dt^2} = X \text{ .... (1),}$$

$$\frac{d^2y}{dt^2} = -\mu y \text{ .... (2),}$$

where X is the required force and μ a constant quantity. Also the equation to the parabola will be

$$y^2 = 4mx \text{ .... (3).}$$

Differentiating this equation,

$$y\frac{dy}{dt} = 2m\frac{dx}{dt},$$

$$\frac{dy^2}{dt^2} + y\frac{d^2y}{dt^2} = 2m\frac{d^2x}{dt^2} = 2mX, \text{ by (1), ...(4).}$$

The integral of (2) is

$$y = C\sin(\mu^{\frac{1}{2}}t + \epsilon).$$

Let V be the velocity of projection; then

$$V = C\mu^{\frac{1}{2}}\cos\epsilon.$$

Also $y = 0$, initially, and therefore

$$0 = C\sin\epsilon.$$

Hence
$$y = \frac{V}{\mu^{\frac{1}{2}}}\sin(\mu^{\frac{1}{2}}t) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5).$$

From (5) we easily see that

$$\frac{dy^2}{dt^2} = V^2 - \mu y^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(6).$$

From (2), (4), (6),

$$2mX = V^2 - 2\mu y^2,$$

$$X = \frac{V^2}{2m} - \frac{\mu}{m}y^2 = \frac{V^2}{2m} - 4\mu x, \text{ by (3)}, \;\ldots\ldots\ldots (7),$$

which determines the required force.

Also, from (1) and (7),

$$\frac{d^2x}{dt^2} = \frac{V^2}{2m} - 4\mu x,$$

$$\frac{d^2}{dt^2}\left(x - \frac{V^2}{8\mu m}\right) + 4\mu\left(x - \frac{V^2}{8\mu m}\right) = 0;$$

the integral of this equation is

$$x - \frac{V^2}{8\mu m} = C\cos(2\mu^{\frac{1}{2}}t + \epsilon):$$

since $x = 0$ and $\frac{dx}{dt} = 0$, initially, this integral is easily reduced to the form

$$x = \frac{V^2}{8\mu m}\operatorname{vers}(2\mu^{\frac{1}{2}}t) \;\ldots\ldots\ldots\ldots\ldots\ldots \;(8).$$

From the expressions (5) and (8) for y and x it appears that the particle oscillates continually in a portion of the parabola cut

off by a double ordinate at a distance $\dfrac{V^2}{4\mu m}$ from the vertex; and that the period of a complete oscillation is $\dfrac{\pi}{\mu^{\frac{1}{2}}}$.

(4) A particle, attracting with a force varying directly as the distance, moves uniformly in a straight line; to determine the motion of another particle situated in the same plane and subject to its influence, the initial cirumstances of the latter particle being given.

Let the initial position of the attracting particle be taken as the origin of co-ordinates: and let x', y', be the co-ordinates of the attracting, and x, y, of the attracted particle at any time t: then the equations of motion will be

$$\frac{d^2x}{dt^2} = \mu^2 (x' - x), \quad \frac{d^2y}{dt^2} = \mu^2 (y' - y),$$

μ^2 denoting the absolute force of attraction.

But, if α, β, denote the resolved parts of the velocity of the attracting particle parallel to the axes of x, y, which are by hypothesis invariable,

$$x' = \alpha t, \quad y' = \beta t,$$

and therefore

$$\frac{d^2x}{dt^2} = \mu^2 (\alpha t - x) \dots\dots\dots\dots (1),$$

$$\frac{d^2y}{dt^2} = \mu^2 (\beta t - y) \dots\dots\dots\dots (2).$$

From the equation (1),

$$\frac{d^2}{dt^2}(x - \alpha t) + \mu^2 (x - \alpha t) = 0.$$

The integral of this equation is

$$x = A \cos (\mu t) + B \sin (\mu t) + \alpha t \dots\dots\dots (3);$$

where A, B, are arbitrary constants. Differentiating we have

$$\frac{dx}{dt} = - A\mu \sin (\mu t) + B\mu \cos (\mu t) + \alpha \dots\dots\dots (4).$$

Let a, m, be the initial values of x, $\frac{dx}{dt}$; then, from (3) and (4),

$$a = A, \quad m = B\mu + \alpha,$$

and therefore, from (3),

$$x = a\cos(\mu t) + \frac{m-\alpha}{\mu}\sin(\mu t) + \alpha t.$$

In precisely the same way, b and n being the initial values of y and $\frac{dy}{dt}$,

$$y = b\cos(\mu t) + \frac{n-\beta}{\mu}\sin(\mu t) + \beta t.$$

(5) A particle is moving in a plane under the action of a force always perpendicular to a line drawn from the particle to a fixed point in the plane: to find the law of the force that the angular velocity of the particle about the point may be constant, and to determine the path described.

Let O be the fixed point in the plane, P the position of the particle at any time t, θ the inclination of OP to a fixed line Ox in the plane of the motion, $OP = r$, $G =$ the force. Then, by formulæ proved in systematic treatises on the motion of particles,

$$\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} = 0 \quad\text{.........................}\quad (1),$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = G \quad\text{.......................}\quad (2).$$

Let $\frac{d\theta}{dt} = \omega$, ω being a constant: then

$$\frac{d^2r}{dt^2} = \omega^2 r,$$

$$r = \alpha e^{\omega t} + \beta e^{-\omega t},$$

α and β being constants.

Hence $$G = 2\omega\frac{dr}{dt} = 2\omega^2(\alpha e^{\omega t} - \beta e^{-\omega t}),$$

and therefore

$$r^2 - \frac{G^2}{4\omega^4} = (\alpha e^{\omega t} + \beta e^{-\omega t})^2 - (\alpha e^{\omega t} - \beta e^{-\omega t})^2 = 4\alpha\beta,$$

$$G = 2\omega^2(r^2 - 4\alpha\beta)^{\frac{1}{2}},$$

which gives the law of the force.

Since $\omega t = \theta$, we have

$$r = \alpha e^{\theta} + \beta e^{-\theta}$$

for the equation to the path.

(6) An imperfectly elastic particle, subject to the action of gravity, is projected from an assigned point in a horizontal plane with a given velocity and in a given direction; to find the velocity of incidence and of reflection, and also the total range with the corresponding time of flight, after the particle has described by rebounding any number of parabolic arcs.

Let e be the elasticity of the particle; u_x the velocity at each end of the x^{th} parabolic arc, and α_x the inclination of the curve at these points to the horizon; t_x the time which elapses before the x^{th} impact; s_x the distance of the point of x^{th} impact from the initial position of the particle.

By the theory of impact we have

$$u_{x+1} \cos \alpha_{x+1} = u_x \cos \alpha_x \qquad (1),$$

$$u_{x+1} \sin \alpha_{x+1} = e u_x \sin \alpha_x \qquad (2);$$

and, by the properties of the motion of projectiles,

$$\Delta t_x = \frac{2}{g} u_{x+1} \sin \alpha_{x+1} \qquad (3),$$

$$\Delta s_x = u_{x+1} \cos \alpha_{x+1} \Delta t_x \qquad (4).$$

From (1) it is evident that

$$u_x \cos \alpha_x = u_1 \cos \alpha_1 \qquad (5),$$

where u_1 is the given velocity and α_1 the given angle of projection.

Again, from (2), putting $u_x \sin \alpha_x = v_x$, we have

$$\Delta v_x = (e-1) v_x;$$

and therefore, integrating,

$$v_x = Ce^x,$$

where C is an arbitrary constant. But $x=1$, $v_x = v_1$, simultaneously; hence $Ce = v_1$, and therefore

$$v_x = v_1 e^{x-1}, \qquad u_x \sin \alpha_x = u_1 \sin \alpha_1 e^{x-1} \qquad (6).$$

From (5) and (6) we get

$$\tan \alpha_x = \tan \alpha_1 . e^{x-1}, \qquad u_x = u_1 \cos \alpha_1 . \{1 + \tan^2 \alpha_1 . e^{2(x-1)}\}^{\frac{1}{2}},$$

by which the circumstances of the projection are determined for each of the parabolic paths.

Again, from (3) and (6),

$$\Delta t_x = \frac{2}{g} u_1 \sin \alpha_1 e^x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

integrating and adding a constant,

$$t_x = \frac{2u_1 \sin \alpha_1}{g} \frac{e^x}{e-1} + C;$$

but $t_0 = 0$, hence

$$0 = \frac{2u_1 \sin \alpha_1}{g} \frac{1}{e-1} + C,$$

and therefore $$t_x = \frac{2u_1 \sin \alpha_1}{g} \frac{e^x - 1}{e-1}.$$

Again, from (4) and (7),

$$\Delta s_x = \frac{2}{g} u_1 \sin \alpha_1 . e^x . u_{x+1} \cos \alpha_{x+1},$$

and therefore, by (5),

$$\Delta s_x = \frac{u_1^2 \sin 2\alpha_1}{g} e^x,$$

whence, integrating and observing that $s_0 = 0$, we shall have

$$s_x = \frac{u_1^2 \sin 2\alpha_1}{g} \frac{1-e^x}{1-e}.$$

Bordoni; *Memorie della Societa Italiana*, Tom. XVII. P. I. p. 191; 1816.

(7) A particle is projected obliquely from a point A at an angle α with the horizon, so as to hit a point B, AB being inclined at an angle β to the horizon; and the velocity of projection is such that with it the particle would describe the straight line AB uniformly in n seconds; to find the time of flight.

$$\text{The required time} = n \frac{\cos \beta}{\cos \alpha}.$$

(8) To find the angle at which a body must be projected from a point in a given inclined plane, in order to impinge upon the plane at right angles; the plane of projection being at right angles to the inclined plane.

If $\alpha =$ the inclination of the given plane to the horizon, the angle which the direction of projection must make with this plane is equal to

$$\tan^{-1}\left(\frac{\cot\alpha}{2}\right).$$

(9) A spherical particle, of which e is the elasticity, is projected with a velocity v at an angle of elevation α, and, at the instant of attaining its greatest altitude, strikes horizontally a similar and equal particle falling downwards with a velocity $\frac{1}{2}v$ at the point of collision; to find the distance of the particles from each other at the end of t seconds after the impact.

$$\text{The distance required } = \tfrac{1}{2}vt\,(1 + 4e^2\cos^2\alpha)^{\frac{1}{2}}.$$

(10) If two particles be projected from the same point, at the same instant, with velocities v, v', and at angles of elevation α, α'; to find the time which elapses between their transits through the other point which is common to both their paths.

$$\text{The required time } = \frac{2}{g}\,\frac{vv'\sin(\alpha \sim \alpha')}{v\cos\alpha + v'\cos\alpha'}.$$

(11) If α be the angle between the two tangents at the extremities of any arc of the parabolic path of a particle acted on by gravity; v, v', the velocities at these two points, and v_1 the velocity at the vertex; to find the time through the arc.

$$\text{The required time } = \frac{vv'\sin\alpha}{gv_1}.$$

(12) A cannon being pointed towards the top of a tower, the ball is seen to strike, after t seconds, a point of the tower on the same horizontal line with the cannon: being reloaded with a different charge, and raised to twice its former angle of elevation, the ball is observed to strike the top of the tower after τ seconds: to find the distance of the tower from the cannon.

The required distance is equal to

$$\tfrac{1}{2}gt^2 \cdot \left(\frac{\tau^2+t^2}{\tau^2-t^2}\right)^{\frac{1}{2}}.$$

(13) If a projectile passes through three points (a, b), (a', b'), (a'', b''); to find the equation connecting these co-ordinates, the point of projection being the origin, and the axis of x being horizontal.

The required equation is

$$\frac{\frac{b}{a}}{(a-a')(a-a'')}+\frac{\frac{b'}{a'}}{(a'-a'')(a'-a)}+\frac{\frac{b''}{a''}}{(a''-a)(a''-a')}=0.$$

(14) Three observers are placed in the line in which the plane of a projectile's motion intersects a horizontal plane, their distances from a given point of this line being a, $a_{,}$, $a_{,,}$, respectively; the greatest angular elevation of the projectile is, for each observer, respectively, $\tan^{-1}\alpha$, $\tan^{-1}\alpha_{,}$, $\tan^{-1}\alpha_{,,}$; to find the greatest height it attains above the plane.

The greatest height is equal to

$$\alpha\alpha_{,}\alpha_{,,} \cdot \frac{a(\alpha_{,}-\alpha_{,,})+a_{,}(\alpha_{,,}-\alpha)+a_{,,}(\alpha-\alpha_{,})}{a^2(\alpha_{,}-\alpha_{,,})+a_{,}^2(\alpha_{,,}-\alpha)+a_{,,}^2(\alpha-\alpha_{,})}.$$

(15) A particle describes an ellipse under the action of a force at right angles to the axis major; to find the force at any point of the path.

Let a, b, be the semiaxes major and minor, y the distance of the particle at any point of its path from the axis major, β the velocity of the particle parallel to the axis major which will remain invariable during the whole motion. Then

$$\text{the force required} = \frac{b^4\beta^2}{a^2y^3}.$$

If $b=a$, or the ellipse become a circle,

$$\text{the force} = \frac{a^2\beta^2}{y^3}.$$

Riccati; *Comment. Bonon.* Tom. IV. p. 149; 1757.
Newton; *Principia*, Lib. I. sect. 2, prop. 8.

(16) If a particle be acted on by a vertical force so as to describe the common catenary, to determine the force and the velocity at any point.

If β = the horizontal velocity of the particle, then, the equation to the catenary being

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

the required force and velocity are respectively equal to $\frac{\beta^2}{c^2} \cdot y$ and $\frac{\beta}{c} \cdot y$.

(17) A particle describes the arc of a cycloid under the action of a force parallel to its base; to find the law of the force.

If the equations to the cycloid be

$$x = a \text{ vers } \theta, \qquad y = a(\theta + \sin\theta),$$

and F, β, denote the force required and the velocity parallel to the axis of the cycloid,

$$\frac{1}{F} = \frac{2a}{\beta^2} \sin\theta \sin^2\frac{\theta}{2}.$$

(18) A particle is projected with a given velocity parallel to a given straight line towards which it is always attracted with a force proportional to its perpendicular distance from it; to determine the position of the particle at any time and the equation to its path.

Let A (fig. 122) be the initial position of the particle; Ox the given straight line; draw yAO at right angles to Ox; let Ox, Oy, be the axes of x, y; P the position of the particle after a time t; let $OM = x$, $PM = y$; $AO = b$, β = the velocity of projection; and let μ^2 be the absolute force of attraction. Then

$$x = \beta t, \quad b \cos\frac{\mu x}{\beta} = y = b\cos(\mu t).$$

Riccati; *Comment. Bonon.* Tom. IV. p. 155; 1757.

(19) A particle is projected from a point $x = 0$, $y = b$, with a velocity β parallel to the axis of x, and is subject to the action

of a force tending towards the axis of x parallel to the axis of y, and varying inversely as the square of the distance; to find the equation to the path of the particle.

Let μ denote the attracting force at a unit of distance; then the equation to the path will be

$$\left(\frac{2\mu}{b}\right)^{\frac{1}{2}}\frac{x}{\beta} = \tfrac{1}{2}b\left\{\pi - \text{vers}^{-1}\frac{2y}{b}\right\} + (by - y^2)^{\frac{1}{2}}.$$

Riccati; *Comment. Bonon.* Tom. IV. p. 159; 1757.

(20) A particle is projected from O (fig. 123) with a given velocity in the direction Oy, and is acted on by a centre of force, which attracts directly as the distance and moves uniformly with a given velocity along Ox at right angles to Oy; to determine the position of the particle when its motion first becomes parallel to Ox.

Let μ^2 denote the absolute force; a the initial distance of the centre of force from O; β the velocity with which the particle is projected, β' the uniform velocity of the centre of force along Ox, and x', y', the co-ordinates of the required position of the particle; then

$$x' = a + \frac{\beta'}{2\mu}(\pi - 2), \quad y' = \frac{\beta}{\mu}.$$

(21) A particle, which is placed at rest initially in a given position, is acted on by two forces, one repulsive and varying as the distance from a given point, and the other constant and acting in parallel lines; to determine the position of the particle at any time and the equation to its path.

Let the centre of the central force be taken as the origin of co-ordinates, and let the directions of the axes be so chosen that the direction of the constant force makes an angle of 45° with each of them. Then, if a, b, be the co-ordinates of the initial position of the particle, μ^2 the absolute force of repulsion, and f the constant force, we shall have, putting $f = 2^{\frac{1}{2}}\mu^2 m$,

$$\frac{x+m}{a+m} = \tfrac{1}{2}(\epsilon^{\mu t} + \epsilon^{-\mu t}) = \frac{y+m}{b+m}.$$

27 (22) Four equal particles, attracting directly as the distance, are fixed in the corners of a square; to find the path of a particle projected from the centre of the square in any direction in the plane of the square.

Let the centre of the square be taken as the origin of co-ordinates, and let the axes be at right angles to the two pairs of opposite sides of the square. Then, if $2m$, $2n$, be the resolved parts of its velocity of projection parallel to the axes of x, y, respectively, and μ^2 the absolute force of attraction of each of the fixed particles, the equation to the path of the free particle will be

$$\frac{x^2}{m^2}+\frac{y^2}{n^2}=\frac{1}{\mu^2}.$$

28 (23) A particle describes a cycloid under the action of a force, which in every position of the body is directed towards the centre of the corresponding generating circle: to find the law of the force and of the motion of the centre of force.

The centre of the generating circle moves uniformly and the force is constant.

Mackenzie and Walton; *Solutions of the Cambridge Problems* for 1854.

(24) The locus of the direction of projection being a plane, and the velocity of projection constant, to find the locus of the trajectories described by bodies acted upon only by gravity.

The required locus is the surface of an elliptic paraboloid, the axis of which is vertical, and the point of projection is the umbilicus of the surface.

(25) A heavy particle, having been projected at a given angle to the inclined plane AB, (fig. 124), proceeds to ascend this plane by bounding in a series of parabolic arcs; to determine the angles of incidence and reflection after any number of impacts.

Let ι be the inclination of the plane AB to the horizon; α_x the angle of reflection in the x^{th} arc, β_{x-1} the angle of incidence in the $(x-1)^{\text{th}}$; and e the elasticity of the particle. Then

$$\tan \alpha_x = \frac{(1-e)\, e^{x-1} \tan \alpha_1}{1 - e - 2(1 - e^{x-1}) \tan \iota \tan \alpha_1} = e \tan \beta_{x-1}.$$

Bordoni; *Memorie della Societa Italiana*, Tom. XVII. P. I. p. 191; 1816.

(26) A ball, of which the elasticity is e, is projected with a velocity V in a direction making an angle $\alpha + \iota$ with the horizon, and rebounds from a plane inclined to the horizon at an angle ι and passing through the point of projection. To determine the relation between R_x, R_{x+1}, R_{x+2}, three consecutive ranges upon the inclined plane after x, $x+1$, $x+2$, rebounds respectively, and to find the sum of all the ranges on the inclined plane before the ball begins to slide down the plane.

If $\cot \beta = (1-e) \cot \iota$, and S denote the sum of the ranges;

$$R_{x+2} - (e + e^2) R_{x+1} + e^3 R_x = 0,$$

$$S = \frac{2V^2 \sin \beta \sin \alpha \cos (\alpha + \beta)}{g \sin \iota \, . \cos^2 \beta}.$$

SECT. 2. *Central Forces.*

Let the force which acts on a particle tend always towards a fixed centre, which we will take as the origin of co-ordinates. Let F denote the force at any distance from the centre; x, y, the co-ordinates of the particle at the end of a time t reckoned from an assigned epoch, r its distance from the centre of force, and θ the inclination of this distance to any fixed line in the plane of x, y. Then, by Maclaurin's Equations, the plane of co-ordinates being identical with the plane of the motion,

$$\frac{d^2x}{dt^2} = -\frac{Fx}{r}, \qquad \frac{d^2y}{dt^2} = -\frac{Fy}{r}.$$

From these equations may be obtained the following formulæ:

$$r^2 d\theta = h dt \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)};$$

$$v = \frac{h}{p} \ldots\ldots\ldots\ldots\ldots\ldots \text{(II)},$$

$$v^2 = h^2 \left\{ \left(\frac{d}{d\theta} \frac{1}{r} \right)^2 + \frac{1}{r^2} \right\} \ldots\ldots\ldots\ldots \text{(III)},$$

$$v^2 = v'^2 - 2\int_{r'}^{r} F dr \text{ .... (IV)},$$

$$F = \frac{h^2}{r^2}\left\{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right\} \text{ .... (V)},$$

$$F = \frac{h^2}{p^3}\frac{dp}{dr} \text{ .... (VI)},$$

$$F = r\frac{d\theta^2}{dt^2} - \frac{d^2 r}{dt^2} \text{ .... (VII)}.$$

In these formulæ h represents twice the area swept out by the radius vector about the centre of force in a unit of time, p the perpendicular from the centre upon the tangent at any point of the orbit, v the velocity of the particle, and v', r', any simultaneous values of v, r. If the central force, instead of being attractive as we have been supposing, be repulsive, we must replace F in these formulæ by $-F$.

The equation (I) shews that the area swept out by the radius vector varies as the time, and either of the equations (II) and (III) that the velocity at any point of the orbit varies inversely as the perpendicular from the centre of force upon the tangent to the orbit at that point: these two propositions were first established by Newton[1]. The equation (IV) shews that the velocity of the particle at any point of its path depends only upon the distance of the point from the centre, the velocity of projection and the prime radius vector, whatever be the course which it may have pursued; the discovery of this proposition is likewise due to Newton[2]. The formula (V), by which the path of the particle may be determined when we know the law of the central force and conversely, Ampère[3] ascribes to Binet. The formula (VI) was communicated without demonstration to John Bernoulli by De Moivre in the year 1705; a proof of the formula was returned to him by Bernoulli in a letter dated Basle, Feb. 1706. The formula (VII) was given much about

[1] *Principia*, Lib. I. Prop. 1.
[2] *Ib.* Lib. I. Prop. 40.
[3] *Annales de Gergonne*, Tom. XX. p. 53.

the same time by Clairaut[1] and by Euler[2], and signifies that the acceleration of the radius vector is equal to the excess of the centrifugal above the attractive force.

(1) To find the law of the force by which a particle may be made to describe the Lemniscata of James Bernoulli, the centre of force coinciding with the node, and to investigate the time of describing one of the ovals.

The polar equation to the Lemniscata is

$$r^2 = a^2 \cos 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)};$$

hence
$$\frac{1}{r} = \frac{1}{a\,(\cos 2\theta)^{\frac{1}{2}}}, \qquad \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{\sin 2\theta}{a\,(\cos 2\theta)^{\frac{3}{2}}},$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{2}{a\,(\cos 2\theta)^{\frac{1}{2}}} + \frac{3\,(\sin 2\theta)^2}{a\,(\cos 2\theta)^{\frac{5}{2}}} = \frac{3}{a\,(\cos 2\theta)^{\frac{5}{2}}} - \frac{1}{a\,(\cos 2\theta)^{\frac{1}{2}}},$$

and therefore

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{a\,(\cos 2\theta)^{\frac{5}{2}}} = \frac{3a^4}{r^5}.$$

Hence, by the formula (V),

$$F = \frac{3a^4h^2}{r^7}.$$

Again, by the formula (I) and the equation (1), we have

$$hdt = r^2 d\theta = a^2 \cos 2\theta \, d\theta,$$

and therefore, if P denote the required periodic time,

$$P = \frac{a^2}{h}\int_{-\frac{1}{4}\pi}^{+\frac{1}{4}\pi} \cos 2\theta \, d\theta = \frac{a^2}{h}.$$

Let μ denote the value of F when $r = 1$; then we have

$$\mu = 3a^4h^2, \qquad h = \frac{\mu^{\frac{1}{2}}}{3^{\frac{1}{2}}a^2}, \qquad P = \frac{3^{\frac{1}{2}}a^4}{\mu^{\frac{1}{2}}}.$$

(2) A particle moves in an equiangular spiral under the action of a force tending towards the pole; to find the law of force and the velocity at any point of the orbit.

[1] *Théorie de la Lune*, p. 2; the first edition of which appeared in 1752, from a MS. sent to St Petersburg in 1750.

[2] *Nov. Comment. Petrop.* 1752, 1753, p. 164.

If β be the invariable angle, r the radius vector, and p the perpendicular from the pole upon the tangent,

$$p = r\sin\beta \qquad\qquad (1).$$

Differentiating with respect to r, we have $\dfrac{dp}{dr} = \sin\beta$, and therefore, from (VI),

$$F = \frac{h^2}{p^3}\sin\beta = \frac{h^2}{r^3\sin^2\beta}\text{, by (1),} \qquad\qquad (2).$$

Let c be the velocity corresponding to a given radius vector r'; then, by (II) and (1),

$$h = cr'\sin\beta.$$

Hence, from (2), $$F = \frac{c^2r'^2}{r^3},$$

and, from (II) and (1),

$$v = \frac{cr'\sin\beta}{r\sin\beta} = \frac{cr'}{r}.$$

(3) A particle describes an equilateral hyperbola round a centre of force situated in the centre; to find the law of the force and the angle which the particle will describe about the centre from the apse in a given time.

The equation to the hyperbola being

$$\left(\frac{1}{r}\right)^2 = \frac{\cos 2\theta}{a^2} \qquad\qquad (1),$$

we have $$\frac{1}{r}\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\sin 2\theta}{a^2} \qquad\qquad (2),$$

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left\{\frac{d}{d\theta}\left(\frac{1}{r}\right)\right\}^2 = -\frac{2\cos 2\theta}{a^2};$$

and therefore, from (2),

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{r^2}{a^4}\sin^2 2\theta = -\frac{2\cos 2\theta}{a^2},$$

and thence, by (1),

$$\frac{1}{r}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{r^2}{a^4} - \frac{1}{r^2} = -\frac{2}{r^2},$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^3}{a^4}.$$

Hence, by the formula (V),

$$F = -\frac{h^2 r}{a^4}.$$

The negative sign shews that the force must be repulsive; let $-\mu$ be the absolute force, that is, the value of F at a unit of distance. Then

$$\mu = \frac{h^2}{a^4}, \quad F = -\mu r.$$

Putting for h its value $a^2\mu^{\frac{1}{2}}$ in the formula (I), and $\dfrac{a^2}{\cos 2\theta}$ for r^2, we get

$$\frac{d\theta}{\cos 2\theta} = \mu^{\frac{1}{2}}dt, \quad \frac{\cos 2\theta\, d\theta}{1-\sin^2 2\theta} = \mu^{\frac{1}{2}}t, \quad \frac{d\sin 2\theta}{1-\sin^2 2\theta} = 2\mu^{\frac{1}{2}}dt;$$

integrating, and supposing the time to be reckoned from apsidal passage, we have

$$\tfrac{1}{2}\log\frac{1+\sin 2\theta}{1-\sin 2\theta} = 2\mu^{\frac{1}{2}}t,$$

whence, writing μ' in place of $4\mu^{\frac{1}{2}}$, we obtain

$$\sin 2\theta = \frac{\epsilon^{\mu' t}-1}{\epsilon^{\mu' t}+1}.$$

(4) A particle is revolving in a parabola about a centre of force in the focus, and, when it arrives at a given distance from the focus, the absolute force is suddenly doubled; to determine the nature of the subsequent path of the particle.

Let $4m$ be the latus rectum of the parabola, r the radius vector at any point, and p the perpendicular from the focus upon the tangent. Then, by the nature of the parabola,

$$\frac{1}{p^2} = \frac{1}{mr}, \quad \frac{2}{p^3}\frac{dp}{dr} = \frac{1}{mr^2},$$

and therefore, by (VI),

$$F = \frac{h^2}{2mr^2}.$$

But, after the absolute force has been doubled, we shall have for the motion

$$F = \frac{h^2}{mr^2},$$

and therefore, by (VI),

$$\frac{h^2}{mr^2} = \frac{h^2}{p^3}\frac{dp}{dr}, \quad \frac{1}{mr^2} = \frac{1}{p^3}\frac{dp}{dr}.$$

Integrating, we have

$$\frac{1}{mr} = C + \frac{1}{2p^2}.$$

Let c be the value of r at the instant when the absolute force is doubled; then, p being then common both to the parabola and to the new path, we have

$$\frac{1}{mc} = C + \frac{1}{2mc}, \quad C = \frac{1}{2mc},$$

and therefore, for the equation to the new path, there is

$$\frac{1}{mr} = \frac{1}{2mc} + \frac{1}{2p^2}, \quad \frac{mc}{p^2} = \frac{2c}{r} - 1,$$

which is the equation to an ellipse, $2c$ being the major and $2(mc)^{\frac{1}{2}}$ the minor axis.

Since the ellipse touches the parabola when $r = c$, the semi-axis major, it follows from the nature of the ellipse that the point of contact is an extremity of the semi-axis minor, and therefore that the axis major of the ellipse is parallel to the tangent at the point $r = c$ of the parabola. But the sine of the angle of inclination of the tangent of the parabola at this point to its axis is $\frac{p}{r} = \frac{m^{\frac{1}{2}}}{r^{\frac{1}{2}}}$, when $r = c$, that is, $= \frac{m^{\frac{1}{2}}}{c^{\frac{1}{2}}}$, and therefore the inclination of the major axis of the ellipse to the axis of the parabola is $\sin^{-1}\left\{\frac{m^{\frac{1}{2}}}{c^{\frac{1}{2}}}\right\}$.

(5) A particle is describing a curve about a certain centre of force, the velocity of the particle varying inversely as the n^{th} power of its distance from the centre of force; to find the law of the force and the equation to the path.

We shall have, μ denoting some constant quantity,

$$v = \frac{\mu}{r^n}.$$

Hence, from (IV), there is

$$\frac{\mu^2}{r^{2n}} = v'^2 - 2\int_{r'}^{r} F dr,$$

and therefore, differentiating,

$$F = \frac{n\mu^2}{r^{2n+1}},$$

which determines the law of the force.

Again, from (III) there is

$$\frac{\mu^2}{r^{2n}} = h^2 \left\{\left(\frac{d}{d\theta}\frac{1}{r}\right)^2 + \frac{1}{r^2}\right\},$$

$$\left(\frac{\mu^2 - h^2 r^{2n-2}}{r^{2n}}\right)^{\frac{1}{2}} = h\frac{d}{d\theta}\frac{1}{r},$$

$$(n-1)\, d\theta = -\frac{dr^{n-1}}{\left(\frac{\mu^2}{h^2} - r^{2n-2}\right)^{\frac{1}{2}}},$$

$$(n-1)\,\theta = C + \cos^{-1}\frac{hr^{n-1}}{\mu}.$$

Suppose $\theta = 0$, when $r = a$; then, k denoting a constant quantity,

$$(n-1)\,\theta = \cos^{-1}(kr^{n-1}) - \cos^{-1}(ka^{n-1}).$$

Riccati; *Comment. Bonon.* Tom. IV. p. 184.

(6) If the force vary as the n^{th} power of the distance, and a particle be projected from an apsidal distance, with a velocity of which the square is equal to $1 - \epsilon$ times the square of the velocity in a circle about the same centre of force with a radius equal to the apsidal distance; to find the equation to the orbit, ϵ being a very small quantity.

Let a be the apsidal distance; then $r = a - x$, where x is a small quantity, because the path of the particle, as is evident from the initial circumstances, will be nearly circular. Then, approximately,

$$\frac{1}{r} = \frac{1}{a-x} = \frac{1}{a}\left(1 + \frac{x}{a}\right), \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{a^2}\frac{d^2x}{d\theta^2}.$$

Also, μ denoting the absolute force of attraction,

$$\frac{Fr^2}{h^2} = \frac{\mu r^{n+2}}{h^2} = \frac{\mu}{h^2}(a-x)^{n+2} = \frac{\mu a^{n+2}}{h^2}\left\{1 - \frac{(n+2)\,x}{a}\right\}.$$

Hence, by the formula (V),

$$\frac{1}{a^2}\frac{d^2x}{d\theta^2} + \frac{x}{a^2} + \frac{1}{a} - \frac{\mu a^{n+2}}{h^2}\left\{1 - \frac{(n+2)x}{a}\right\} = 0,$$

$$\frac{d^2x}{d\theta^2} + \left\{1 + (n+2)\frac{\mu a^{n+3}}{h^2}\right\} x + a - \frac{\mu a^{n+4}}{h^2} = 0 \text{ ....(1)},$$

Let V be the velocity of projection, and v the velocity in a circle about the same centre of force with a radius a; then

$$V^2 = (1-\epsilon)\, v^2 = (1-\epsilon)\, \mu a^{n+1} \text{ ....(2)}.$$

But, by the formula (II),

$$h^2 = a^2V^2,$$

because the motion is initially at right angles to the radius vector, and a, V, are the initial values of the radius vector and of the velocity. Hence, from (2),

$$h^2 = \mu(1-\epsilon)\, a^{n+3}, \quad \frac{\mu}{h^2} = \frac{1}{(1-\epsilon)\, a^{n+3}},$$

and therefore, from (1), the product of ϵ and x being neglected,

$$\frac{d^2x}{d\theta^2} + (n+3)\, x - \epsilon a = 0.$$

Multiplying by $2\frac{dx}{d\theta}$ and integrating,

$$\frac{dx^2}{d\theta^2} + (n+3)\, x^2 - 2\epsilon a x = 0,$$

no constant being added because $\frac{dx}{d\theta}$ is by hypothesis equal to zero when $x = 0$; hence

$$d\theta = \frac{dx}{(n+3)^{\frac{1}{2}}\left(\frac{2\epsilon a}{n+3}\, x - x^2\right)^{\frac{1}{2}}}.$$

Integrating and considering zero to be the initial value of θ,

$$(n+3)^{\frac{1}{2}}\, \theta = \text{vers}^{-1}\frac{(n+3)\, x}{\epsilon a},$$

whence for the polar equation to the curve we have

$$r = a - \frac{\epsilon a}{n+3}\, \text{vers}\left\{(n+3)^{\frac{1}{2}}\theta\right\} \text{ ....(3)}.$$

The general condition for an apsidal distance is evidently that $\frac{dr}{d\theta}$ shall be equal to zero: differentiating the equation (3), we get for the determination of apses,

$$\frac{dr}{d\theta} = -\frac{\epsilon a}{(n+3)^{\frac{1}{2}}} \sin \{(n+3)^{\frac{1}{2}}\theta\} = 0.$$

Hence $$(n+3)^{\frac{1}{2}}\theta = \lambda\pi,$$

where λ is any integer: let θ', θ'', be the values of θ for two consecutive apsidal distances; then

$$(n+3)^{\frac{1}{2}}\theta' = \lambda\pi, \quad (n+3)^{\frac{1}{2}}\theta'' = (\lambda+1)\pi,$$

and therefore, ϕ denoting the angle between any two consecutive apses,

$$\phi = \theta'' - \theta' = \frac{\pi}{(n+3)^{\frac{1}{2}}}.$$

(7) A particle, attached to one end of a slightly extensible string which is extended to its natural length and has its other end fixed, is projected at right angles to it; to determine the extension of the string at any time and the path of the particle, the string resting on a smooth horizontal plane and being indefinitely fine.

Let a, c, denote the initial length of the string and the initial velocity of the particle. Let $r = a + x$ represent the length of the string at the end of any time t, the corresponding angular co-ordinate being θ, and the initial position of the string the prime radius vector. Let m denote the mass of the particle, and p the tension necessary to stretch the string to a length $a + \beta$. Then, the extension being, by Hooke's law, proportional to the tension, we have for the tension at the time t the expression $\frac{px}{\beta}$. Hence, by the formula (VII), we have

$$\frac{d^2r}{dt^2} = r\frac{d\theta^2}{dt^2} - \frac{px}{m\beta},$$

and therefore, by (I),

$$\frac{d^2r}{dt^2} = \frac{h^2}{r^3} - \frac{px}{m\beta}.$$

But from the circumstances of the projection of the particle it is clear by the formula (II) that $h = ac$; hence

$$\frac{d^2r}{dt^2} = \frac{a^2c^2}{r^3} - \frac{px}{m\beta},$$

$$\frac{d^2x}{dt^2} = \frac{a^2c^2}{(a+x)^3} - \frac{px}{m\beta}.$$

Multiplying by $2\dfrac{dx}{dt}$ and integrating,

$$\frac{dx^2}{dt^2} = C - \frac{a^2c^2}{(a+x)^2} - \frac{px^2}{m\beta}:$$

but, initially, $x = 0$, $\dfrac{dx}{dt} = 0$; hence $C = c^2$, and therefore

$$\frac{dx^2}{dt^2} = c^2 - \frac{a^2c^2}{(a+x)^2} - \frac{px^2}{m\beta}$$

$$= c^2 - c^2\left(1 - \frac{2x}{a}\right) - \frac{px^2}{m\beta}$$

$$= \frac{2c^2x}{a} - \frac{px^2}{m\beta} \quad \text{.......................... (1)},$$

where small quantities of the first order only are retained, $\dfrac{x}{\beta}\,.\,x$ being of the first order because x and β are of the same order. Extracting the root, we have

$$\frac{dt}{dx} = \left(\frac{m\beta}{p}\right)^{\frac{1}{2}}\left(2\,\frac{m\beta c^2}{ap}\,x - x^2\right)^{-\frac{1}{2}};$$

whence, integrating and bearing in mind that $x = 0$ when $t = 0$,

$$t = \left(\frac{m\beta}{p}\right)^{\frac{1}{2}} \text{vers}^{-1}\left(\frac{pax}{mc^2\beta}\right),$$

or $$x = \frac{mc^2\beta}{pa}\,\text{vers}\left\{\left(\frac{p}{m\beta}\right)^{\frac{1}{2}} t\right\},$$

which gives the extension of the string at any time during the motion.

We proceed now to the determination of the equation to the path of the particle. From (I) and (1) there is

$$\frac{dx^2}{d\theta^2} = \frac{r^4}{h^2}\left(\frac{2c^2x}{a} - \frac{px^2}{m\beta}\right)$$

$$= \frac{(a+x)^4}{a^2c^2}\left(\frac{2c^2x}{a} - \frac{px^2}{m\beta}\right)$$

$$= 2ax - \frac{pa^2x^2}{mc^2\beta}, \text{ approximately,}$$

$$= \frac{pa^2}{mc^2\beta}\left(2\frac{mc^2\beta}{pa}x - x^2\right),$$

$$\frac{d\theta}{dx} = \left(\frac{mc^2\beta}{pa^2}\right)^{\frac{1}{2}}\left(2\frac{mc^2\beta}{pa}x - x^2\right)^{-\frac{1}{2}}.$$

Integrating, and remembering that $\theta = 0$, $x = 0$, simultaneously,

$$\theta = \left(\frac{mc^2\beta}{pa^2}\right)^{\frac{1}{2}}\text{vers}^{-1}\frac{pax}{mc^2\beta},$$

$$r = a + x = a + \frac{mc^2\beta}{pa}\text{ vers}\left\{\left(\frac{pa^2}{mc^2\beta}\right)^{\frac{1}{2}}\theta\right\} \text{............ (2),}$$

which is the polar equation to the orbit.

From this equation we have, for the determination of apses,

$$\frac{dx}{d\theta} \propto \sin\left\{\left(\frac{pa^2}{mc^2\beta}\right)^{\frac{1}{2}}\theta\right\} = 0:$$

hence the angle between consecutive apses is

$$\frac{\pi c}{a}\left(\frac{m\beta}{p}\right)^{\frac{1}{2}}.$$

In the solution of this problem James Bernoulli takes as the approximate equation of motion

$$\frac{d^2r}{dt^2} = \frac{c^2}{\rho} - \frac{px}{m\beta},$$

where ρ denotes the radius of curvature at any point of the orbit. The difference, however, between $\frac{c^2}{\rho}$ and $\frac{h^2}{r^3}$ is of the first order of small quantities, and therefore his approximate equation is erroneous. Instead of the equation (2) he gets

$$r = a + \frac{mc^2\beta}{pa}\text{ vers}\left\{\left(\frac{pa^2}{2mc^2\beta}\right)^{\frac{1}{2}}\theta\right\},$$

which makes the angle between the apses greater than it should be in the ratio of $2^{\frac{1}{2}}$ to 1.

James Bernoulli; *Nov. Act. Acad. Petrop.* 1783, p. 213.

(8) A particle moves in a spiral, of which the equation is $r = a\left(\sec\frac{\theta}{n}\right)^n$, about a centre of force in the pole; to find the law of the force.

If μ denote the absolute force, then

$$F = \frac{\mu}{r^{\frac{3n-2}{n}}}.$$

(9) To find the law of force by which the Cissoid of Diocles may be described about a centre of force in the cusp.

If θ be the angle which the radius vector r makes with the axis, then

$$F \propto \frac{\operatorname{cosec}^2\theta}{r^3}.$$

(10) A particle revolves in a circle about a centre of force situated at a point in its circumference; to determine the force and the velocity at any point of the path.

If μ denote the absolute force,

$$F = \frac{\mu}{r^5}, \qquad v^2 = \frac{\mu}{2r^4}.$$

Newton; *Princip.* Lib. I. Prop. 7. Riccati; *Comment. Bonon.* Tom. IV. p. 175.

(11) A particle is moving about a centre of force, its velocity at each point of its path varying inversely as its distance from the centre of force; to determine the path of the particle.

The path will be a logarithmic spiral.

Riccati; *Ib.* p. 184.

(12) A body, attracted towards a centre of force which varies inversely as the square of the distance, is projected with a velocity equal to the velocity in a circle at an equal distance, and in a direction making an angle of 45° with the radius vector: to find the magnitude and position of the orbit described.

The orbit will be an ellipse, the point of projection being an extremity of the minor axis and the centre of force a focus. The

prime radius vector being a, the axis major $= 2a$, the axis minor $= a\sqrt{2}$, the excentricity $= \dfrac{1}{\sqrt{2}}$.

(13) If the force vary as the seventh power of the distance, and a particle be projected from an apse with a velocity which is to the velocity in a circle at the same distance as 1 to $\sqrt{3}$; to find the equation to the curve described.

If the apsidal distance a be taken as the prime radius vector, the equation to the curve described is

$$r^2 = a^2 \cos 2\theta.$$

(14) A particle, projected in a given direction with a given velocity and attracted towards a given centre of force, has its velocity at every point to the velocity in a circle at the same distance as 1 to $\sqrt{2}$: to find the orbit described and the law of force.

Let S (fig. 125) be the centre of force, P the point of projection; let $SP = a$, $\beta =$ the angle between PS and the direction PT of projection.

Draw PA at right angles to SP and equal to $a \cot \beta$: join SA. The orbit described is a circle of which $SA = a \operatorname{cosec} \beta$ is the diameter; the law of force being that of the inverse fifth power.

(15) A particle moves in an equilateral hyperbola about a centre of force in the centre; to find the locus of the points to which the particle must move from the curve under the action of the force to acquire the velocity in the curve.

If a be the semi-axis of the equilateral hyperbola in which the particle is moving, the required locus will also be an equilateral hyperbola having a semi-axis equal to $2^{\frac{1}{2}} . a$, the centres of the two hyperbolas being coincident and their axes in the same straight lines.

(16) An indefinitely fine elastic string, extended to its natural length, is fastened at one end and has a particle of matter attached to the other; the particle is projected at right angles to the string with an angular velocity such that, if it were

revolving in a circle with this angular velocity, the length of the string must have been stretched to twice its natural length; to find the path which the particle will ultimately describe after an indefinite number of revolutions.

If a be the natural length of the string, the particle will ultimately move in a circle the value of the radius r of which is a root of the equation

$$\frac{a^2}{r^2} = 2\frac{r-a}{r+a}.$$

(17) In a curve described by a body by the action of a central force, the angle between the radius vector and the tangent varies as the time: to find the curve and the law of force.

If F = the force, and β, h, ω, be certain constants, then

$$F = \beta^2 . h . e^{-\frac{\omega r^2}{h}} . \frac{\omega r^2 + h}{r^3},$$

and the differential equation to the path is

$$\theta = \int \frac{dr}{r\left(\beta^2 . e^{-\frac{\omega r^2}{h}} - 1\right)^{\frac{1}{2}}}.$$

(18) A particle describes a circular orbit about a centre of force at the centre of the circle; to find the condition that the form of the orbit may be stable or unstable.

If P = the central force, u = the reciprocal of the radius vector, and $\frac{1}{a}$ = the radius of the circle, the form of the orbit will be stable or unstable accordingly as

$$\frac{d \log P}{d \log u}, \quad \text{when } u = a,$$

is less or not less than 3.

(19) A particle, attracted towards a point by a force equal to $\frac{r}{m^2} + \frac{h^2}{r^3}$, is projected from an apse at the distance $m^{\frac{1}{2}} h^{\frac{1}{2}}$, h being twice the area described in a unit of time; to find the polar

equation to the orbit, and the time of describing any angle about the centre of force.

If θ be the angle described about the centre of force in any time t after the projection,

$$r^2 = \frac{mh}{1+\theta^2}, \qquad t = m \tan^{-1}\theta.$$

(20) At a distance a from a centre of force, a particle is projected at an angle of 45° to the distance, and with a velocity which is to that in a circle at the same distance as $2^{\frac{1}{2}}$ to $3^{\frac{1}{2}}$; the central force at any distance r is equal to $\dfrac{2a^2n}{r^5} + \dfrac{n}{r^3}$; to find the equation to the orbit.

If the angular co-ordinate be estimated from the initial radius vector, the equation to the orbit will be

$$r = a(1-\theta).$$

(21) A particle, acted on by a central force varying as any function of the distance, is projected at an apse with a velocity nearly equal to that requisite for a circular orbit; to find the distance, from the centre of force, of the apse at which the particle next arrives.

Let the force at any distance r be equal to $\dfrac{1}{r^2}\phi\left(\dfrac{1}{r}\right)$, where $\phi\left(\dfrac{1}{r}\right)$ is any function of $\dfrac{1}{r}$; let a be the initial distance of the particle from the centre of force, a' the distance of the apse at which it next arrives, and let the velocity of projection be to the velocity in a circle about the same centre as 1 to $1+m$; then

$$a' = a\left\{1 + \frac{2m\phi\left(\frac{1}{a}\right)}{\phi\left(\frac{1}{a}\right) - \frac{1}{a}\phi'\left(\frac{1}{a}\right)}\right\}.$$

SECT. 3. *Tangential and Normal Resolutions.*

The method of the solution of the general problem of the curvilinear motion of a particle in one plane, by the principle of the tangential and normal resolutions, is expressed by the equations

$$v\frac{dv}{ds} = T \text{..........................} (A),$$

$$\frac{v^2}{\rho} = N \text{..........................} (B),$$

where v denotes the velocity at any point of the path, ds an element of the curve, ρ the radius of curvature, T the sum of the resolved parts of the forces which act on the particle estimated in the direction of its motion, and N the sum of the resolved parts along the normal on the concave side of the curve in the neighbourhood of the particle.

(1) A particle is projected with a given velocity and in a given direction, and is acted upon by a constant force in parallel lines; to determine the path of the particle.

Let the axis of x be taken so as to pass through the initial place of the particle, and let the axis of y be taken parallel to the constant force which acts towards the axis of x. Let f denote the constant force. Then, the tangential resolved part being $-f\frac{dy}{ds}$, and the normal one being $f\frac{dx}{ds}$, we have for the motion of the particle,

$$v\frac{dv}{ds} = -f\frac{dy}{ds} \text{..........................} (1),$$

$$\frac{v^2}{\rho} = f\frac{dx}{ds} \text{..........................} (2).$$

Integrating (1), $\qquad v^2 = C - 2fy.$

Let V be the initial velocity; then, y being zero initially, $V^2 = C$, and therefore

$$v^2 = V^2 - 2fy.$$

Hence, substituting this expression for v^2 in (2),

$$\frac{1}{\rho}(V^2 - 2fy) = f\frac{dx}{ds};$$

but
$$\rho = -\frac{\dfrac{ds^3}{dx^3}}{\dfrac{d^2y}{dx^2}};$$

hence
$$-\frac{d^2y}{dx^2}(V^2 - 2fy) = f\frac{ds^2}{dx^2} = f\left(1 + \frac{dy^2}{dx^2}\right),$$

$$(V^2 - 2fy)\frac{d}{dx}\left(1 + \frac{dy^2}{dx^2}\right) - \left(1 + \frac{dy^2}{dx^2}\right)\frac{d}{dx}(V^2 - 2fy) = 0.$$

Integrating, we have
$$C\left(1 + \frac{dy^2}{dx^2}\right) = V^2 - 2fy,$$

where C is an arbitrary constant. Let β be the angle which the direction of projection makes with the axis of x; then

$$C(1 + \tan^2\beta) = V^2:$$

hence
$$V^2\left(1 + \frac{dy^2}{dx^2}\right) = \sec^2\beta\,(V^2 - 2fy),$$

$$V^2\frac{dy^2}{dx^2} = V^2\tan^2\beta - 2f\sec^2\beta\, y,$$

$$Vdy = (V^2\tan^2\beta - 2f\sec^2\beta\, y)^{\frac{1}{2}}\, dx;$$

whence, by integration,

$$C - V(V^2\tan^2\beta - 2f\sec^2\beta\, y)^{\frac{1}{2}} = f\sec^2\beta\, x.$$

But $x = 0$, $y = 0$, simultaneously; hence

$$C - V^2\tan\beta = 0,$$

and therefore

$$V^2\tan\beta - V(V^2\tan^2\beta - 2f\sec^2\beta\, y)^{\frac{1}{2}} = f\sec^2\beta\, x.$$

Clearing the equation of radicals, and simplifying,

$$y = \tan\beta \,.\, x - \frac{f\sec^2\beta}{2V^2}x^2.$$

Euler; *Mechan.* Tom. I. p. 232.

(2) A particle, always acted on by a force in parallel lines, describes a given curve; to determine the nature of the force, the velocity and direction of projection being given.

Let the force be represented by Y, which we will suppose to

act towards the axis of x parallel to the axis of y. The equations of motion will be

$$v\frac{dv}{ds} = -Y\frac{dy}{ds} \quad \text{.........................} \quad (1),$$

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} \quad \text{.........................} \quad (2).$$

Eliminating Y, we have

$$\frac{1}{v}\frac{dv}{ds} = -\frac{1}{\rho}\frac{dy}{dx} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds^3}{dx^3}},$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds^2}{dx^2}} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{1+\frac{dy^2}{dx^2}}.$$

Integrating, we get

$$\log v = C + \tfrac{1}{2}\log\left(1+\frac{dy^2}{dx^2}\right) = C + \log\frac{ds}{dx}.$$

Let V denote the initial velocity, and β the angle which the direction of projection makes with the axis of x; then

$$\log V = C + \log \sec\beta,$$

and therefore
$$\log\frac{v}{V} = \log\frac{\frac{ds}{dx}}{\sec\beta},$$

$$v = V\cos\beta\,\frac{ds}{dx}.$$

Substituting this value of v in (2), we have

$$Y = \frac{V^2\cos^2\beta}{\rho}\,\frac{ds^3}{dx^3}.$$

Euler; *Mechan.* Tom. I. p. 240.

(3) A particle describes a given curve about a centre of force; to determine the motion of the particle and the law of the force.

Let APB (fig. 126) be the path of the particle, S the centre of force; P the position of the particle at any time; PT a tangent at the point P, and SY perpendicular to PT. Let F

denote the force along PS, and ϕ the angle SPT; then, the direction of the motion at P being towards B,

$$v\frac{dv}{ds} = -F\cos\phi \quad\text{......................} (1),$$

$$\frac{v^2}{\rho} = \quad F\sin\phi \quad\text{......................} (2).$$

Now, since $$ds\cos\phi = dr,$$

and $$\rho\sin\phi = \frac{rdr}{dp}\sin\phi = p\frac{dr}{dp},$$

where p denotes SY, we have, by (1) and (2),

$$vdv = -Fdr \quad\text{..........................} (3),$$

$$v^2 = \quad Fp\frac{dr}{dp} \quad\text{......................} (4).$$

Eliminating F between (3) and (4),

$$\frac{dv}{v} = -\frac{dp}{p}, \quad \log v = C - \log p.$$

Let V, P, be the initial values of v, p; then

$$\log V = C - \log P,$$

and therefore $$\log\frac{v}{V} = \log\frac{P}{p}, \quad v = \frac{V.P}{p} \quad\text{.................} (5).$$

Again, if t denote the time of the motion,

$$\frac{ds}{dt} = v = \frac{V.P}{p}, \text{ by (5)},$$

$$pds = VPdt;$$

but pds is equal to dh', where h' represents twice the area swept out by the radius vector in its motion from some assigned position; hence $$dh' = VPdt, \quad h' = VPt \quad\text{...................} (6),$$

the area being supposed to commence with the time.

Again, by (2), we have

$$F = \frac{v^2}{\rho\sin\phi} = \frac{v^2}{p}\frac{dp}{dr} = \frac{V^2P^2}{p^3}\frac{dp}{dr}, \text{ by (5)}, \quad\text{............} (7).$$

Suppose now that h represents twice the area swept out in a unit of time; then, since, by (6), h is equal to VP, we have, by (6), (5), (7),

$$h' = ht \qquad\qquad (8),$$

$$v = \frac{h}{p} \qquad\qquad (9),$$

$$F = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2 r}{\rho p^3} \qquad\qquad (10).$$

The formulæ (8) and (9) were given by Newton[1]. The formula (10) was discovered by De Moivre in the year 1705, by whom it was communicated without demonstration to John Bernoulli. A proof of the formula was obtained by Bernoulli and forwarded to De Moivre in a letter dated Basle, Feb. 16, 1706. Demonstrations were afterwards given by Keill[2], and by Hermann[3]. See De Moivre's *Miscell. Analyt.* Lib. VIII., and John Bernoulli, *Opera*, Tom. I. p. 477.

Integrating the equation (3) we get another expression for the velocity,

$$v^2 = V^2 - 2\int_R^r F dr,$$

where R denotes the prime radius vector. This result shews that the velocity of the particle depends only upon its distance from the centre of force, and not upon the path described; a theorem proved by Newton[4].

Euler; *Mechan.* Tom. I. p. 240.

(4) Bodies are projected with the same velocity and from the same point but in different directions, and describe curves about a centre of force: to find the locus of the centres of the circles of curvature to the different orbits, at the point of projection.

The locus is a straight line cutting at right angles the distance between the point of projection and the centre of force.

[1] *Principia*, Lib. I. Prop. 1.
[2] *Phil. Trans. Num.* 317; 1708.
[3] *Phoronomia*, p. 70.
[4] *Princip.* Lib. I. Prop. 40.

SECT. 4. *Motion in Resisting Media.*

(1) A particle acted on by gravity is projected in a uniform medium, of which the resistance varies as the velocity, with a given velocity and at a given angle of inclination to the horizon; to find after what interval of time the particle will arrive at its greatest altitude.

Let k be the resistance for a unit of velocity, u the volocity and α the angle of projection, and let y be the height through which the particle has ascended at the end of the time t. Then

$$\frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\frac{dy}{ds} = -g - k\frac{dy}{dt},$$

$$\log\left(g + k\frac{dy}{dt}\right) = C - kt;$$

but $\frac{dy}{dt} = u \sin\alpha$ when $t = 0$; hence

$$\log(g + ku\sin\alpha) = C,$$

and therefore

$$\log\frac{g + ku\sin\alpha}{g + k\frac{dy}{dt}} = kt.$$

When y is a maximum, $\frac{dy}{dt} = 0$, and therefore the required value of t will be equal to

$$\frac{1}{k}\log\left(1 + \frac{k}{g}u\sin\alpha\right).$$

(2) A particle moving in a resisting medium is acted on by a given force in parallel lines; to find the resistance that any proposed curve may be described, and conversely.

Let the positions of the particle be referred to two rectangular axes Ox, Oy, (fig. 127); let $OM = x$, $PM = y$, $AP = s$; P being the position of the particle at any time, and APB the curve of its motion; also let Y denote the accelerating force at P parallel to Oy, v the velocity of the particle, and R the resistance of the medium.

Then, by the equations of tangential and normal resolution given in the preceding section, we have

$$v\frac{dv}{ds} = -R + Y\frac{dy}{ds} \dots\dots\dots\dots\dots\dots (1),$$

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} \dots\dots\dots\dots\dots\dots\dots\dots (2);$$

where ρ denotes the radius of curvature at P. But

$$\rho = \frac{\frac{ds^3}{dx^3}}{\frac{d^2y}{dx^2}};$$

hence, from (2),

$$v^2 = \frac{\frac{ds^2}{dx^2}}{\frac{d^2y}{dx^2}} Y;$$

differentiating, we have, since $\dfrac{ds^2}{dx^2} = 1 + \dfrac{dy^2}{dx^2}$,

$$v\frac{dv}{dx} = Y\frac{dy}{dx} + \frac{1}{2}\frac{ds^2}{dx^2}\frac{d}{dx}\left\{\frac{Y}{\frac{d^2y}{dx^2}}\right\},$$

$$v\frac{dv}{ds} = Y\frac{dy}{ds} + \frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left\{\frac{Y}{\frac{d^2y}{dx^2}}\right\};$$

hence, from (1), $$R = -\frac{1}{2}\frac{ds}{dx}\frac{d}{dx}\left\{\frac{Y}{\frac{d^2y}{dx^2}}\right\},$$

which gives the resistance if the curve be given, and conversely.

The solution of this problem, which Newton had given in the first edition of the *Principia*, involved certain errors, which at the suggestion of John Bernoulli were afterwards corrected.

COR. If the resistance vary as the square of the velocity for a uniform density; then, Q denoting the density generally, we have

$$R = Qv^2 = Q\rho \frac{dx}{ds} Y, \text{ by (2)},$$

and therefore
$$Q = \frac{R}{Y} \frac{\frac{d^2y}{dx^2}}{\frac{ds^2}{dx^2}}$$

$$= -\frac{1}{2} \frac{dx}{ds} \frac{\frac{d^2y}{dx^2}}{Y} \frac{d}{dx} \left\{ \frac{Y}{\frac{d^2y}{dx^2}} \right\}$$

$$= -\frac{1}{2} \frac{dx}{ds} \frac{d}{dx} \log \left\{ \frac{Y}{\frac{d^2y}{dx^2}} \right\},$$

which gives the density at any point within the medium; or, if the density be given, determines the curve.

COR. It is evident that $\rho \frac{dx}{ds}$ is equal to half the chord of curvature at P parallel to Oy, or in the direction of the force Y; let q denote this chord of curvature. Then

$$v^2 = 2Y \cdot \tfrac{1}{4}q;$$

and therefore the space due to the velocity, supposing the force to continue constant, is equal to one-fourth of the chord of curvature.

Newton; *Princip.* Lib. II. Prop. 10. John Bernoulli; *Act. Erudit. Lips.* 1713; *Opera*, Tom. I. p. 514.

(3) A particle moves in a resisting medium under the action of a given force always tending towards a fixed centre; to determine the law of resistance when the path of the particle is given, and conversely.

Let APB (fig. 128) be the path of the particle, S the centre of force; let $AP = s$, $SP = r$, $p =$ the perpendicular from S upon the tangent at P, $v =$ the velocity at P; let ρ be the radius of curvature, P the central force, and R the resistance of the medium.

Then, by the equations of tangential and normal resolution, we have

$$v\frac{dv}{ds} = -R - P\frac{dr}{ds} \qquad \text{.....................} (1),$$

$$\frac{v^2}{\rho} = \frac{p}{r}P \qquad \text{.............................} (2).$$

Since ρ is equal to $r\dfrac{dr}{dp}$, we have, from (2),

$$v^2 = p\frac{dr}{dp}P \qquad \text{............................} (3);$$

and therefore, differentiating with respect to s,

$$\begin{aligned} v\frac{dv}{ds} &= \frac{1}{2}\frac{d}{ds}\left(p\frac{dr}{dp}P\right) \\ &= \frac{1}{2}\frac{d}{ds}\left\{\left(p^3\frac{dr}{dp}P\right)\frac{1}{p^2}\right\} \\ &= -\frac{1}{p^3}\frac{dp}{ds}\left(p^3\frac{dr}{dp}P\right) + \frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right) \\ &= -\frac{dr}{ds}P + \frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right). \end{aligned}$$

Hence, by substituting this value of $v\dfrac{dv}{ds}$ in (1), we have

$$R = -\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right) \qquad \text{.................} (4);$$

which determines the law of resistance when the curve is known, and conversely.

Cor. Supposing the resistance to vary as the density of the medium multiplied by the square of the velocity of the particle, we have, Q denoting the density,

$$R = Qv^2 = Qp\frac{dr}{dp}P, \text{ by (3)},$$

and therefore, by (4),

$$Q = -\frac{1}{2}\frac{\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right)}{p^3\frac{dr}{dp}P} = -\frac{1}{2}\frac{d}{ds}\log\left(p^3\frac{dr}{dp}P\right) \qquad \text{..............} (5),$$

which determines the law of the density when the curve is given, and conversely.

COR. From the equation (2) we have

$$v^2 = \frac{p}{r} \rho P = \tfrac{1}{2} q P = 2 (\tfrac{1}{4} q) P,$$

where q denotes the chord of curvature through S. This result shews that the velocity at any point of the curve is that due to falling in vacuum towards the centre of force, continued constant, through a quarter of the chord of curvature.

COR. From (4) we have

$$p^3 \frac{dr}{dp} P = h^2 \epsilon^{-2\int Q ds}, \quad P = \frac{h^2 \frac{dp}{dr}}{p^3} \epsilon^{-2\int Q ds},$$

where h is some constant quantity. This formula gives the central force when the law of the density and the orbit are given. It is easily shewn that, if $\angle ASP = \theta$,

$$\frac{h^2}{p^3} \frac{dp}{dr} = \frac{h^2}{r^2} \left\{ \frac{1}{r} + \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \right\},$$

and therefore $$P = \frac{h^2}{r^2} \left\{ \frac{1}{r} + \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \right\} \epsilon^{-2\int Q ds}.$$

If $Q = 0$, we get $$P = \frac{h^2}{r^2} \left\{ \frac{1}{r} + \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \right\},$$

which is Binet's formula for the central force in vacuum.

Newton; *Principia*, Lib. II. Prop. 17, 18; John Bernoulli; *Opera*, Tom. IV. p. 347. Euler; *Mechan.* Tom. I. p. 428 et sq., p. 451 et sq.

(4) A particle is projected with a given velocity in a uniform medium, in which the resistance varies as the square of the velocity; the particle is acted on by gravity, and the direction of its projection makes a very small angle with the horizon; to determine approximately the equation to the portion of the path which lies above the horizontal plane passing through the point of projection.

Let Ox and Oy (fig. 129) be the horizontal and vertical axes of x and y, O being the point of projection; P the position of

the particle at any time; let $OM=x$, $PM=y$, $v=$ the velocity at P, $s=$ the arc OP, $k=$ the resistance for a unit of velocity; then, by the tangential and normal resolutions,

$$v\frac{dv}{ds}=-kv^2-g\frac{dy}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$v^2=g\frac{dx}{ds}\rho=-g\frac{1+p^2}{q} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $p=\frac{dy}{dx}$ and $q=\frac{dp}{dx}$. Hence, eliminating v between these two equations, we have

$$\frac{d}{ds}\frac{1+p^2}{q}+2k\frac{1+p^2}{q}-2\frac{dy}{ds}=0,$$

$$\frac{d}{dx}\frac{1+p^2}{q}+2k\frac{1+p^2}{q}\frac{ds}{dx}-2p=0,$$

$$\frac{d}{dx}\log\frac{1+p^2}{q}+2k\frac{ds}{dx}-\frac{2pq}{1+p^2}=0.$$

Integrating, we get

$$\log\frac{1+p^2}{q}+2ks+\log C-\log(1+p^2)=0,$$

C being some constant quantity;

$$\log\frac{C}{q}+2ks=0,\quad q=Ce^{2ks} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let u be the velocity, and α the angle of projection; then, initially, by (3) and (2),

$$q=C,\quad u^2=-g\frac{1+\tan^2\alpha}{q},$$

and therefore $$C=-\frac{g}{u^2\cos^2\alpha};$$

hence, by (3), $$q=-\frac{g}{u^2\cos^2\alpha}e^{2ks}:$$

but, the angle of projection being small, we may neglect all powers of p beyond the first, and therefore

$$s=\int_0^x(1+p^2)^{\frac{1}{2}}dx=\int_0^x dx=x \text{ nearly};$$

hence $$q = -\frac{g}{u^2 \cos^2 \alpha} e^{2kx} \text{ nearly.}$$

Multiplying by dx, and integrating,

$$p = C - \frac{g}{2ku^2 \cos^2 \alpha} e^{2kx};$$

but $p = \tan \alpha$ when $x = 0$; hence

$$\tan \alpha = C - \frac{g}{2ku^2 \cos^2 \alpha};$$

and therefore $$p = \tan \alpha + \frac{g}{2ku^2 \cos^2 \alpha}(1 - e^{2kx}).$$

Integrating again,

$$y = C + x \tan \alpha + \frac{g}{2ku^2 \cos^2 \alpha}\left(x - \frac{1}{2k} e^{2kx}\right);$$

but $x = 0$, $y = 0$, simultaneously; hence

$$0 = C - \frac{g}{4k^2u^2 \cos^2 \alpha},$$

and therefore $$y = x \tan \alpha + \frac{g}{4k^2u^2 \cos^2 \alpha}(1 + 2kx - e^{2kx}).$$

Moreau; *Journal de l'Ecole Polytech.* Cahier XI. p. 215.

The general problem of the path of a projectile in a uniform resisting medium where the resistance varies as the square of the velocity, was proposed by Keill as a trial of skill to John Bernoulli, by whom the challenge was received in February 1718. Keill, trusting to the complexity of the analysis, which had probably deterred Newton from attempting any regular solution of the problem in the second book of the *Principia*, was in hopes that the exertions of Bernoulli would prove unsuccessful. Bernoulli, however, having expeditiously effected a solution, not only of Keill's problem, but likewise of the more general one where the resistance varies as the n^{th} power of the velocity, expressed a determination not to publish his investigation until he had received intimation that his antagonist had himself been able to solve his own problem. He gave Keill till the following September to exercise his talents, declaring that if he received by that time no satisfactory communication, he should feel him-

self entitled to question the ability of his adversary. At the request of a mutual friend, Bernoulli consented to extend the interval to the first of November. It turned out, however, that Keill was unable to obtain a solution. At length Nicholas Bernoulli, Professor of Mathematics at Padua, communicated to John Bernoulli a solution of Keill's problem, which the author afterwards extended to the more general one. Finally, on the 17th of November, information was received by John Bernoulli, from Brook Taylor, to the effect that he had obtained a solution. John Bernoulli published his own analysis, together with that of his nephew Nicholas, in the *Acta Erudit. Lips.* 1719 mai. p. 216; see also his *Opera*, Tom. II. p. 393. For further information on this celebrated problem, the reader may avail himself of the labours of Euler[1], Borda[2], Legendre[3], Templehoff[4], and Moreau[5].

(5) A particle moves in a semicircle acted on by gravity, in a medium where the resistance varies as the density, and the square of the velocity; to find the law of the resistance and density.

Let OA, OB, (fig. 130), be horizontal and vertical radii of the semicircle, OAx and OBy being the axes of x and y; let a = the radius of the circle, $OM = x$, and gravity $= g$. Then

$$R = \frac{3}{2}\frac{gx}{a}, \quad v^2 = g(a^2 - x^2)^{\frac{1}{2}}, \quad Q = \frac{3x}{2a}(a^2 - x^2)^{-\frac{1}{2}}.$$

Newton; *Princip.* Lib. II. Prop. 10. Ex. 1. John Bernoulli; *Act. Erudit. Lips.* 1713. Euler; *Mechan.* Tom. II. p. 392.

(6) A particle acted on by gravity moves in a parabola of any order; to find the law of resistance.

Let the equation to the parabola ABC (fig. 130) be $y = b - cx^n$, Oy being vertical; then

$$R = \frac{(n-2)(1 + n^2c^2x^{2n-2})^{\frac{1}{2}} g}{2n(n-1)cx^{n-1}}.$$

[1] *Mém. de l'Acad. de Berlin*, 1753.
[2] *Ib.* 1769.
[3] *Ib.* 1782.
[4] *Ib.* 1788—89.
[5] *Journal de l'Ecole Polytech.* Cahier XI. p. 204.

(7) A particle moves in an hyperbola of any order, having one of its asymptotes vertical; to find the law of the density, the resistance varying as the density into the square of the velocity.

Let APB (fig. 131) denote the path of the particle, Oy the vertical asymptote being taken as the axis of y, and Ox, which is horizontal, as the axis of x; then, if the equation to the hyperbola be

$$y = ax + \frac{\beta^{n+1}}{x^n},$$

we shall have
$$Q = \frac{\frac{1}{2}(n+2)\,x^n}{\{x^{2n+2} + (ax^{n+1} - n\beta^{n+1})^2\}^{\frac{1}{2}}}.$$

Euler; *Mechan.* Tom. II. p. 400.

(8) A particle moves in a circle about a centre of force in the circumference, the force being attractive and varying as any power of the distance; to determine the resistance of the medium and the law of the density, supposing the resistance to be equal to the product of the density and the square of the velocity.

Let P (fig. 132) be the position of the particle at any time, its motion taking place towards S the centre of force; let C be the centre of the circle; let $SP = r$, $\angle PSA = \theta$, the central force $= \mu r^n$; then

$$R = \frac{1}{4}\mu(5+n)\,r^n \sin\theta, \quad Q = \frac{1}{2}(n+5)\frac{\sin\theta}{r}.$$

(9) A particle moves in an equiangular spiral about a centre of force in the pole which varies as any power of the distance from the pole; to find the law of the resistance and of the density of the medium, the resistance being considered equal to the product of the density and the square of the velocity.

If α be the constant angle, r the radius vector at any point, μr^n the attractive central force, and the particle be so moving as to approach the centre of force;

$$R = \frac{1}{2}\mu(n+3)\,r^n \cos\alpha, \quad Q = \frac{1}{2}(n+3)\frac{\cos\alpha}{r}.$$

Newton; *Princip.* Lib. II. Prop. 15, 16. John Bernoulli; *Opera*, Tom. IV. p. 350.

(10) A particle moves in the circumference of a circle about a centre of force in the centre; the resistance of the medium in which the motion takes place is constant; to determine the law of the force, the velocity at any time of the motion, and the time which elapses, as well as the space which is described, before the particle is reduced to rest.

Let β be the initial velocity of the particle, a the radius of the circle, c the constant value of the resistance, s the arc described from the beginning of the motion, P the central force; then

$$v^2 = \beta^2 - 2cs, \qquad P = \frac{1}{a}(\beta^2 - 2cs);$$

when the particle is reduced to rest,

$$s = \frac{\beta^2}{2c}, \qquad t = \frac{\beta}{c}.$$

(11) A particle is moving along the curve of an equiangular spiral so as to be approaching the pole; the motion takes place in a medium where the resistance varies as any power of the distance from the pole; to find the law of the central attractive force in the pole.

Let α be the constant angle, β the initial velocity, a the initial distance, kr^n the resistance at a distance r, P the required force; then

$$P = \frac{(n+3)\, a^2\beta^2 \cos\alpha + 2k\,(r^{n+3} - a^{n+3})}{(n+3)\, r^3 \cos\alpha}.$$

Euler; *Mechan.* Tom. I. p. 442.

(12) A particle is projected with a velocity u, and at an inclination α to the horizon, in a uniform medium where the resistance varies as the velocity; to determine the time which elapses before the direction of the motion is inclined to the horizon at an angle β.

If k represent the resistance for a unit of velocity, t will be found from the equation

$$g \cos\beta\,(\epsilon^{kt} - 1) = ku \sin(\alpha - \beta).$$

(13) Two particles, subject to the action of gravity, are simultaneously projected at equal angles of inclination to the horizon,

and with equal velocities, the one in vacuum and the other in a medium where the resistance varies as the velocity; to determine a relation between the times in which the particles describe two arcs so related to each other that the tangents at their extremities shall make equal angles with the horizon.

If k denote the resistance of the medium for a unit of velocity, and t_1, t_2, denote corresponding times in vacuum and in the medium; then

$$e^{kt_2} = 1 + kt_1.$$

(14) Having given the co-ordinates of the highest point of the path described by a particle acted on by gravity and projected in vacuum at a known angle of inclination to the horizon; to find the decrements of these co-ordinates when the particle is projected in a rare medium in which the resistance varies as the velocity.

Let a, b, be the given co-ordinates, k the resistance for a unit of velocity, and β the angle of projection; then

$$\delta a = -k\left(\frac{a^3 \tan\beta}{g}\right)^{\frac{1}{2}}, \quad \delta b = -k\left(\frac{8b^3}{9g}\right)^{\frac{1}{2}}.$$

SECT. 5. *Impossible Mechanics.*

The determination of the motion of a material particle in a plane XOY, (fig. 133), under the action of assigned forces in that plane, depends upon the integration of two simultaneous differential equations

$$\frac{d^2x}{dt^2} = \phi(x, y), \quad \frac{d^2y}{dt^2} = \chi(x, y) \ldots\ldots\ldots\ldots\ldots (1),$$

where x, y, are the co-ordinates of the position of the particle, at the end of any time t, referred, we will suppose, to rectangular axes OX, OY, and $\phi(x, y)$, $\chi(x, y)$, are certain functions of x, y, depending upon the nature of the forces. Suppose that, for certain particular forms of the functions, having effected the integrations and determined the arbitrary constants from the

initial circumstances of the motion, we are enabled to obtain the relations

$$x = \phi_1(t), \qquad y = \chi_1(t) \ldots\ldots\ldots\ldots (2),$$

where $\phi_1(t)$, $\chi_1(t)$, represent certain functions of t. In the determination of these relations is generally supposed to consist the complete solution of the problem of the motion of the particle. There are particular cases, however, in which they cease, after the lapse of a certain time, to correspond to the physical motion of the particle. It occasionally happens that, as t increases indefinitely from zero by continuous gradation, one, or sometimes both of the functions $\phi_1(t)$, $\chi_1(t)$, after remaining possible up to a certain magnitude of t, at length assumes impossible values.

Before proceeding to the consideration of the mechanical circumstances connected with this anomaly, it will be necessary to make a few observations on the geometrical signification of impossible values of x and y.

By the theory of impossible quantities we know that for all values of x and y, possible or impossible, we may assume

$$x = (\cos\theta + -^{\frac{1}{2}} \sin\theta)\, \alpha,$$
$$y = (\cos\phi + -^{\frac{1}{2}} \sin\phi)\, \beta,$$

where α, β, θ, ϕ, are all possible and positive quantities. Suppose now that $\theta = 2\lambda\pi$, where λ is an integer; then we have $x = \alpha$; also, if $\theta = (2\lambda + 1)\pi$, where λ is still supposed to be integral, then $x = -\alpha$. It is evident then that whatever geometrical interpretation may be given to the general form of the value of x, it must be such that, when θ is any even multiple of π, x may denote a distance α estimated along the axis OX in the positive direction; and that, when θ is any odd multiple of π, x may denote a distance α estimated negatively from O, that is, in the direction XO. Similar remarks, *mutatis mutandis*, are evidently applicable to the interpretation of the values of y. These are the only conditions to which we are subject in our selection of a geometrical interpretation of impossible values of the variables. From what has been said then, it appears that the conventional significations of the signs $+$ and $-$, leave that of the more general sign $\cos\theta + -^{\frac{1}{2}} \sin\theta$ in a great measure arbitrary.

In fact

$$x = (\cos\theta + -^{\frac{1}{2}} \sin\theta)\,\alpha$$

may be taken to represent any line OA of a length α drawn from O at an inclination θ to the direction OX; thus the locus of the extremity A will be a circle described with a radius $\alpha \sin\theta$ about a centre in OX, at a distance $\alpha \cos\theta$ from O, the plane of the circle being at right angles to OX. In like manner

$$y = (\cos\phi + -^{\frac{1}{2}} \sin\phi)\,\beta$$

may be taken to represent a straight line $OB = \beta$, inclined at an angle ϕ to OY. Complete the parallelogram $OAPB$; then P will be the geometrical point corresponding to the values of x and y. From this construction it is clear that P is not limited to a particular point, but that it may assume an infinite number of different positions.

In order that, for every pair of values of x and y, the point P may receive a definite position, it will be necessary to adopt, for the construction of the conjugate axes OA, OB, some general law which shall be consistent with the circular loci of the points A, B. For instance, we might assume A and B always to coincide with the intersections of their circular loci with the plane XOY; or, drawing OZ at right angles to the plane XOY, with the intersections of these loci with the planes XOZ, YOZ, or, in fact, with any assigned planes whatever passing through OX, OY. It is evident that the locus described by the point P in accordance with the equations (2), will vary, as far as impossible values of the variables are concerned, with variations in the law of constructing the conjugate axes.

Now, reverting to the consideration of the mechanical value of the equations (2), it is evident that, so long as they perfectly represent the physical motion of the particle, there can be no indeterminateness in their geometrical interpretation, it being impossible for a particle to describe any but one path in accordance with given forces and a given projection. We will proceed to ascertain whether it be possible to adopt such a system of construction for the conjugate axes, as to render the geometrical locus of the equations (2) coincident with the

physical path of the particle under the circumstances of the problem. Let μ, ν, denote the inclinations of the planes AOX, BOY, to the plane XOY. Then, since the particle can never deviate from the plane XOY under the circumstances of the problem, we shall have, by restricting the construction of the axes OA, OB, to the fulfilment of this condition, a relation

$$F(\alpha, \beta, \theta, \phi, \mu, \nu) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also, the effective accelerating forces on the point P parallel to OX, OY, must (when x and y become impossible) continue to coincide with the physical law of the forces. Hence we shall have two more requisite relations,

$$G(\alpha, \beta, \theta, \phi, \mu, \nu) = 0, \quad H(\alpha, \beta, \theta, \phi, \mu, \nu) = 0 \ldots\ldots (4).$$

Now α, β, θ, ϕ are each known functions of t; hence, between the relation (3) and the two relations (4), we may eliminate μ, ν, and thus we shall get an equation involving t and known quantities. Since, then, t is restricted to particular values, it is evident that the equations (2) cannot, for any law of constructing the axes, be enabled to represent the physical motion of the particle for impossible values of the variables.

Whenever, then, x or y assumes an impossible value for any value of t, we must conclude that the physical motion of the particle cannot be represented by any single pair of equations (2). It will be necessary, on arriving at such a critical value of t, again to revert to the equations (1), to integrate them anew, and to determine the values of the arbitrary constants in accordance with the values which x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, have acquired at the conclusion of the preceding stage of the motion. It will be necessary to repeat this process whenever t makes x or y impossible. Corresponding to each integration there will be a new equation in x and y, resulting from the elimination of t between the corresponding pair of equations (2). A portion only of each of the curves belonging to the equations in x and y will be described by the particle, which will therefore pursue a path consisting of fragments of a series of distinct curves.

(1) A particle, placed in the plane XOY, is attracted towards the axes OX and OY by forces varying inversely as the cubes of the distances; to determine the motion of the particle.

This problem gives rise to the differential equations

$$\frac{d^2x}{dt^2} = -\frac{m^4}{x^3}, \quad \frac{d^2y}{dt^2} = -\frac{n^4}{y^3} \dots\dots\dots\dots (1).$$

Integrating these equations, we have

$$\frac{dx^2}{dt^2} = A + \frac{m^4}{x^2}, \quad \frac{dy^2}{dt^2} = B + \frac{n^4}{y^2},$$

A, B, being arbitrary constants. Now, initially, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both equal to zero, and therefore, if a, b, denote the initial values of x, y, we have

$$0 = A + \frac{m^4}{a^2}, \quad 0 = B + \frac{n^4}{b^2}.$$

Hence $$\frac{dx^2}{dt^2} = m^4\left(\frac{1}{x^2} - \frac{1}{a^2}\right), \quad \frac{dy^2}{dt^2} = n^4\left(\frac{1}{y^2} - \frac{1}{b^2}\right) \dots\dots\dots (2).$$

Now, since the particle will move towards OY, $\frac{dx}{dt}$ will be negative, and thus

$$\frac{dx}{dt} = -m^2\,\frac{(a^2 - x^2)^{\frac{1}{2}}}{ax}; \quad \frac{axdx}{(a^2 - x^2)^{\frac{1}{2}}} = -m^2 dt;$$

and therefore $$A' - a\,(a^2 - x^2)^{\frac{1}{2}} = -m^2 t,$$

where A' is an arbitrary constant; but $x = a$ when $t = 0$, and therefore $A' = 0$: hence

$$a\,(a^2 - x^2)^{\frac{1}{2}} = m^2 t$$

$$\left.\begin{aligned} x^2 &= a^2 - \frac{m^4 t^2}{a^2}: \\ y^2 &= b^2 - \frac{n^4 t^2}{b^2}. \end{aligned}\right\} \dots\dots\dots\dots (3).$$

similarly

As soon as t becomes greater than $\frac{a^2}{m^2}$, it is evident, from the former of the equations (3), that x becomes impossible; and, when t becomes greater than $\frac{b^2}{n^2}$, it appears in like manner from

the latter that y becomes impossible. We will commence with the consideration of the motion parallel to the axis of x. When x becomes impossible, the former of the equations (3) can no longer represent the physical motion of the particle parallel to OX, and we must revert to the equation

$$\frac{d^2x}{dt^2} = -\frac{m^4}{x^3},$$

where x is now to be estimated in the negative direction.

Integrating, and adding an arbitrary constant A', we have

$$\frac{dx^2}{dt^2} = A' + \frac{m^4}{x^2} \text{ .... } (4).$$

But, in the beginning of the second stage of the motion, and at the end of the first, the velocity is the same. Hence, by (2) and (4),

$$m^4 \left\{ \frac{1}{(x=0)^2} - \frac{1}{a^2} \right\} = A' + \frac{m^4}{(x=0)^2}, \quad A' = -\frac{m^4}{a^2},$$

and therefore, from (4),

$$\frac{dx^2}{dt^2} = m^4 \left(\frac{1}{x^2} - \frac{1}{a^2} \right).$$

Extracting the square root, we obtain

$$m^2 dt = + \frac{ax\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

or

$$m^2 dt = - \frac{ax\,dx}{(a^2 - x^2)^{\frac{1}{2}}}.$$

Integrating these two equations,

$$m^2 t + C_1 = -a\,(a^2 - x^2)^{\frac{1}{2}},$$

$$m^2 t + C_2 = +a\,(a^2 - x^2)^{\frac{1}{2}}.$$

The former of these equations corresponds to the motion from the axis of y, and the latter to the motion which afterwards takes place towards it. To determine C_1, we have $x = 0$ when $t = \frac{a^2}{m^2}$, which gives $C_1 = -2a^2$; and thus, for the first part of the motion,

$$m^2 t - 2a^2 = -a\,(a^2 - x^2)^{\frac{1}{2}}.$$

When $t=\frac{2a^2}{m^2}$, then $x=a$, and the motion of return begins which is defined by the second of the equations; and $C_2=-2a^2$. Thus both integrals are comprehended by the equation

$$a^2(a^2-x^2)=(m^2t-2a^2)^2.$$

After a time $\frac{3a^2}{m^2}$ it is evident from this equation that x becomes impossible, and then the third period of the motion of the particle takes place. It is evident that the particle will again cross the axis of y, and perform on the other side of it, parallel to the axis of x, a motion exactly similar to that of the second stage of the motion. Similarly for the fourth, fifth, &c. stages of the motion parallel to the axis of x.

From the preceding conclusions it is manifest that, dividing the time from its zero value into a series of intervals of which the first is equal to $\frac{a^2}{m^2}$ and all the following ones to $\frac{2a^2}{m^2}$, the motion of the particle at right angles to OY will be represented for each interval respectively by the following equations,

$$a^2\{a^2-(+x)^2\}=m^4t^2,$$
$$a^2\{a^2-(-x)^2\}=(m^2t-2a^2)^2,$$
$$a^2\{a^2-(+x)^2\}=(m^2t-4a^2)^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

the general formula for the p^{th} interval being

$$a^2\{a^2-(-^{p-1}x)^2\}=\{m^2t-2(p-1)a^2\}^2 \ldots\ldots\ldots\ldots (5).$$

In like manner, for the motion of the particle parallel to the axis of y, if we divide the time into intervals of which the first is equal to $\frac{b^2}{n^2}$ and the rest to $\frac{2b^2}{n^2}$, we shall have for the motion in the q^{th} interval,

$$b^2\{b^2-(-^{q-1}y)^2\}=\{n^2t-2(q-1)b^2\}^2 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Eliminating t between (5) and (6), we obtain as a general

formula for the equations to the series of curves of which portions are successively traversed by the particle,

$$an^2\left[\{a^2-(-^{p-1}x)^2\}^{\frac{1}{2}}+2\,(p-1)\,a\right]=bm^2\left[\{b^2-(-^{q-1}y)^2\}^{\frac{1}{2}}+2\,(q-1)\,b\right].$$

As t keeps continuously increasing from zero, we must keep putting for p and q integral values next greater respectively than

$$\frac{t+\dfrac{a^2}{m^2}}{\dfrac{2a^2}{m^2}}=\frac{m^2t+a^2}{2a^2},$$

and
$$\frac{t+\dfrac{b^2}{n^2}}{\dfrac{2b^2}{n^2}}=\frac{n^2t+b^2}{2b^2}.$$

Thus the formula will give us the series of equations representing the successive curves. The intersections of the successive curves, if we pay attention to the signs $-^{p-1}$ and $-^{q-1}$, will constitute the limits of the portions of each which the particle actually describes.

(2) To investigate the path of a particle the law of the motion of which is expressed by the integrals of the equations

$$\frac{d^2x}{dt^2}=-\frac{m^4}{x^3},\quad \frac{d^2y}{dt^2}=-\frac{n^4}{y^3}\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

corrected in accordance with the conditions that initially $x=a$, $y=b$, $\dfrac{dx}{dt}=0$, $\dfrac{dy}{dt}=0$; the conjugate axes OA, OB, being supposed to be always at right angles to OY, OX, respectively.

This problem will, as we know, correspond to a path very different, after the lapse of a certain time, from the course pursued by the particle in the preceding problem. Integrating the equations (1), as in the first stage of the motion in the preceding problem, we have

$$x^2=a^2-\frac{m^4t^2}{a^2},\quad y^2=b^2-\frac{n^4t^2}{b^2}\;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Assume, as the most general forms of x and y,

$$x = (\cos\theta + -^{\frac{1}{2}}\sin\theta)\,\alpha, \qquad y = (\cos\phi + -^{\frac{1}{2}}\sin\phi)\,\beta;$$

then, from the former of the equations (2),

$$(\cos 2\theta + -^{\frac{1}{2}}\sin 2\theta)\,\alpha^2 = a^2 - \frac{m^4 t^2}{a^2};$$

equating the coefficients of like affections, we get

$$\alpha^2\cos 2\theta = a^2 - \frac{m^4t^2}{a^2} \quad\ldots\ldots\ldots\ldots\ldots(3),$$

$$\sin 2\theta = 0 \quad\ldots\ldots\ldots\ldots\ldots(4).$$

From (4) we see that

$$2\theta = \lambda\pi,$$

where λ is some integer, and therefore, by (3),

$$\alpha^2\cos\lambda\pi = a^2 - \frac{m^4t^2}{a^2}.$$

If t be less than $\frac{a^2}{m^2}$, $a^2 - \frac{m^4t^2}{a^2}$ is positive, and therefore $\cos\lambda\pi$ must be positive; and, if t be greater than $\frac{a^2}{m^2}$, $a^2 - \frac{m^4t^2}{a^2}$ is negative, and therefore $\cos\lambda\pi$ must be negative. Hence, if t be less than $\frac{a^2}{m^2}$,

$$\alpha^2 = a^2 - \frac{m^4t^2}{a^2}, \quad \theta = \mu\pi;$$

and, if t be greater than $\frac{a^2}{m^2}$,

$$\alpha^2 = \frac{m^4t^2}{a^2} - a^2, \qquad \theta = \tfrac{1}{2}(2\mu+1)\,\pi;$$

μ being an integral number.

Similarly, if t be less than $\frac{b^2}{n^2}$,

$$\beta^2 = b^2 - \frac{n^4t^2}{b^2}, \qquad \phi = \mu'\pi,$$

and, if t be greater than $\frac{b^2}{n^2}$,

$$\beta^2 = \frac{n^4t^2}{b^2} - b^2, \qquad \phi = \tfrac{1}{2}(2\mu'+1)\,\pi.$$

Suppose now that $\dfrac{a^2}{m^2}$ is less than $\dfrac{b^2}{n^2}$; then, so long as t is less than $\dfrac{a^2}{m^2}$,

$$\left.\begin{aligned} \alpha^2 &= a^2 - \frac{m^4t^2}{a^2}, & \beta^2 &= b^2 - \frac{n^4t^2}{b^2}, \\ \theta &= \mu\pi, & \phi &= \mu'\pi, \end{aligned}\right\} \text{.............(5)};$$

while t is greater than $\dfrac{a^2}{m^2}$, but less than $\dfrac{b^2}{n^2}$,

$$\left.\begin{aligned} \alpha^2 &= \frac{m^4t^2}{a^2} - a^2, & \beta^2 &= b^2 - \frac{n^4t^2}{b^2}, \\ \theta &= \tfrac{1}{2}(2\mu + 1)\pi, & \phi &= \mu'\pi, \end{aligned}\right\} \text{.............(6)};$$

and, when t is greater than $\dfrac{b^2}{n^2}$,

$$\left.\begin{aligned} \alpha^2 &= \frac{m^4t^2}{a^2} - a^2, & \beta^2 &= \frac{n^4t^2}{b^2} - b^2, \\ \theta &= \tfrac{1}{2}(2\mu + 1), & \phi &= \tfrac{1}{2}(2\mu' + 1), \end{aligned}\right\} \text{.............(7)}.$$

Suppose now that x', y', z', denote rectangular co-ordinates of the particle; x', y', being estimated along OX, OY, and z' along OZ at right angles to the plane XOY. Then it will easily be seen that

$$x' = \alpha \cos \theta, \quad y' = \beta \cos \phi, \quad z' = \alpha \sin \theta + \beta \sin \phi.$$

In the case of the equations (5),

$$x' = \alpha \cos \mu\pi, \quad y' = \beta \cos \mu'\pi, \quad z' = 0,$$

and therefore

$$\left.\begin{aligned} &x'^2 = a^2 - \frac{m^4t^2}{a^2}, \qquad y'^2 = b^2 - \frac{n^4t^2}{b^2}, \\ &\frac{a^2x'^2}{m^4} - \frac{b^2y'^2}{n^4} = -\left(\frac{b^4}{n^4} - \frac{a^4}{m^4}\right), \quad z' = 0, \end{aligned}\right\} \text{.............(8)}.$$

In the case of the equations (6),

$$x' = 0, \quad y' = \beta \cos \mu'\pi, \quad z' = \alpha \cos \mu\pi,$$

and therefore

$$\left.\begin{aligned} &y'^2 = b^2 - \frac{n^4t^2}{b^2}, \qquad z'^2 = a^2 - \frac{m^4t^2}{a^2}, \\ &\frac{b^2y'^2}{n^4} + \frac{a^2z'^2}{m^4} = \frac{b^4}{n^4} - \frac{a^4}{m^4}, \qquad x' = 0, \end{aligned}\right\} \text{.............(9)}.$$

In the case of the equations (7),

$$x'=0, \quad y'=0, \quad z'=\left(\frac{m^4t^2}{a^2}-a^2\right)^{\frac{1}{2}}+\left(\frac{n^4t^2}{b^2}-b^2\right)^{\frac{1}{2}} \ldots\ldots (10).$$

The equations (8) shew that the particle moves from its initial position to the axis of y' in the arc of an hyperbola, the time of the motion being $\frac{a^2}{m^2}$. The equations (9) shew that, on arriving at the axis of y', the particle subsequently moves in the arc of an ellipse to the axis of z', in a time equal to $\frac{b^2}{n^2}-\frac{a^2}{m^2}$. And the equations (10) shew that, after arriving at the axis of z', it perpetually ascends this axis, with which its path for the future entirely coincides. The position of the particle at any assigned time in each of the three discontinuous portions of its path, is given by the corresponding pair of relations between the variables and the time.

CHAPTER IV.

CONSTRAINED MOTION OF A PARTICLE.

SECT. 1. *Motion of a particle within smooth immoveable Tubes.*

LET a particle, under the action of any forces in one plane, move within an indefinitely thin curvilinear tube APB, (fig. 134). Let x, y, be the co-ordinates of the place P of the particle, after a time t from the commencement of the motion; and let $AP = s$, where A is some assigned point in the tube. Let X, Y, represent the resolved parts of the accelerating force acting on the particle parallel to the axes Ox, Oy, and S the resolved part along the tangent to the curve APB at P. Then the equation for the motion of the particle will be

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} = S \quad \text{.................(A)},$$

or, by integration, v denoting the velocity of the particle at the point P,

$$v^2 \text{ or } \frac{ds^2}{dt^2} = 2\int(Xdx + Ydy) + C = 2\int Sds + C \quad \text{.......(B)},$$

where C is an arbitrary constant, introduced by the integration, which may be determined if we know the initial velocity and the initial position of the particle.

If the force acting on the particle be a central force; then, P representing its intensity at a distance r, we have, taking the centre of force as the origin of x, y,

$$Xdx + Ydy = Pdr,$$

and the formulæ (A), (B), become

$$\frac{d^2s}{dt^2} = P\frac{dr}{ds} \quad \text{..........................(C)},$$

$$v^2 \text{ or } \frac{ds^2}{dt^2} = 2\int Pdr + C \quad \text{.................(D)}.$$

(1) A particle, acted on by gravity, descends from rest down a given circular arc, the tangent to which at the lowest point is horizontal; to compare the initial accelerating force estimated along the curve with that which would correspond to motion down the chord, when the arc is indefinitely diminished.

Let A (fig. 135) be the lowest point of the arc, C the centre of the circle, P the position of the particle at any time, T the intersection of the tangent PT with the vertical line CA produced. Draw PM horizontally, and join AP, CP. Let $F=$ the accelerating force at P down the arc PA, $f=$ that down the chord PA, $\angle CTP=\alpha$, $\angle CAP=\alpha'$, $CA=a$, $AM=x$, $PM=y$; $g=$ the force of gravity.

Then $$F=g\cos\alpha,\quad f=g\cos\alpha',$$

and therefore $$\frac{F}{f}=\frac{\cos\alpha}{\cos\alpha'}.$$

But, by the nature of the circle,

$$\cos\alpha=\cos\angle\, CPM=\frac{y}{a}.$$

Also, observing that $y^2=2ax-x^2$, we get

$$\cos\alpha'=\frac{AM}{AP}=\frac{x}{(x^2+y^2)^{\frac{1}{2}}}=\left(\frac{x}{2a}\right)^{\frac{1}{2}}.$$

Hence $$\frac{F}{f}=\frac{y}{a}\left(\frac{2a}{x}\right)^{\frac{1}{2}}=\frac{(2ax-x^2)^{\frac{1}{2}}}{a}\left(\frac{2a}{x}\right)^{\frac{1}{2}}=\left\{\frac{2}{a}(2a-x)\right\}^{\frac{1}{2}};$$

and therefore, when $x=0$, we have, for the required ratio,

$$F=2f.$$

Saurin; *Mémoires de l'Académie des Sciences de Paris*, 1724, p. 70. Louville; *Ib.* p. 128. Courtivron, *Ib.* 1744, p. 384.

(2) The highest point of the circumference of a circle in a vertical plane is connected by means of a chord with some other point in the curve; to determine the time in which a particle, under the action of gravity, will fall down this chord.

Let AB (fig. 136) be the vertical diameter through the highest point A of the circle; AC the chord down which the particle descends. Join BC, and let P be the position of the

particle after a time t from the commencement of its motion. Let $AP = s$, $AB = 2a$, $\angle BAC = \alpha$, $AC = l$. Then, the resolved part of g, the force of gravity, along AC, being $g \cos \alpha$, we have, for the motion of the particle,

$$\frac{d^2s}{dt^2} = g \cos \alpha.$$

Integrating, we get

$$\frac{ds}{dt} = gt \cos \alpha + C;$$

but $\dfrac{ds}{dt} = 0$, $t = 0$, simultaneously; hence $C = 0$; and therefore

$$\frac{ds}{dt} = gt \cos \alpha.$$

Integrating again, and observing that $s = 0$ when $t = 0$, we have

$$s = \tfrac{1}{2}gt^2 \cos \alpha.$$

Let T denote the whole time of descent down AC; then

$$l = \tfrac{1}{2}gT^2 \cos \alpha;$$

but, from the geometry, $l = 2a \cos \alpha$;

hence $$2a \cos \alpha = \tfrac{1}{2}gT^2 \cos \alpha,$$

and therefore $$T = 2\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

This result, being independent of the inclination of the chord to the vertical, shews that the descents down all such chords are performed in the same time; a proposition established by Galileo.

Wolff; *Elementa Matheseos Universæ*, Tom. II. p. 58.

(3) From one extremity of the horizontal diameter of a circle in a vertical plane, two chords are drawn subtending angles α, 2α, at the centre; given that the time down the latter chord is n times as great as that down the former, to find the value of α.

Let β, β', be the inclinations of the two chords to the horizon; l, l', their lengths, and t, t', their times of description. Then, as in the preceding problem, it will be found that

$$l = \tfrac{1}{2}gt^2 \sin \beta, \qquad l' = \tfrac{1}{2}gt'^2 \sin \beta',$$

and therefore $$\frac{l'}{l} = \frac{t'^2}{t^2}\,\frac{\sin \beta'}{\sin \beta} = n^2\,\frac{\sin \beta'}{\sin \beta}.$$

But from the geometry it is evident that

$$l = 2a \cos\beta, \quad l' = 2a \cos\beta';$$

hence $$\frac{\cos\beta'}{\cos\beta} = n^2 \frac{\sin\beta'}{\sin\beta}, \quad \tan\beta = n^2 \tan\beta';$$

but it is clear that

$$2\beta = \pi - \alpha, \quad 2\beta' = \pi - 2\alpha,$$

hence $$\cot\tfrac{1}{2}\alpha = n^2 \cot\alpha,$$

$$\frac{2\tan\frac{1}{2}\alpha}{1 - \tan^2\frac{1}{2}\alpha} = n^2 \tan\tfrac{1}{2}\alpha,$$

$$\frac{1 + \cos\alpha}{\cos\alpha} = n^2,$$

and therefore $$\cos\alpha = \frac{1}{n^2 - 1}, \quad \alpha = \cos^{-1}\frac{1}{n^2 - 1}.$$

(4) A particle is placed anywhere within a thin rectilinear tube, and is acted on by a force tending always towards a fixed centre, and varying directly as the distance; to find the time of an oscillation.

Let x be the distance of the particle, at any time t, from its position of equilibrium, r its corresponding distance from the centre of force, μ^2 the absolute force of attraction. Then, $\mu^2 r$ being the central force at the time t,

$$\frac{d^2x}{dt^2} = -\mu^2 r \frac{x}{r} = -\mu^2 x:$$

the integral of this equation is

$$x = A \cos(\mu t + \epsilon) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where A and ϵ are arbitrary constants.

Let a be the initial value of x; then, from (1),

$$a = A \cos\epsilon:$$

also, $\frac{dx}{dt}$ being initially zero, we have, from (1),

$$0 = A \sin\epsilon:$$

hence, substituting for $A \cos \epsilon$ and $A \sin \epsilon$ their values, the equation (1) is reduced to

$$x = a \cos \mu t.$$

Now, when x acquires its greatest negative value, $\mu t = \pi$; hence, T denoting the period of a complete oscillation, we have

$$T = \frac{\pi}{\mu}.$$

Euler; *Mechan.* Tom. II. p. 91. Cor. 4.

(5) A particle is constrained to move in a straight line, and is attached to one end of an indefinitely fine elastic string, the other end of which is fixed at a distance from the straight line equal to the unstretched length of the string; to find the time of a small oscillation.

Let a = the natural length of the string, m = the mass of the particle; s = its distance at any time t from its position of equilibrium, T = the tension of the string, and l = its length at the same time. Then, for the motion of the particle,

$$m \frac{d^2 s}{dt^2} = -\frac{Ts}{l} \text{.........................} (1).$$

Again, by Hooke's law of extension,

$$l = a(1 + \epsilon T) \text{.......................} (2),$$

where ϵ is a constant quantity depending upon the extensibility of the string.

But, s being a small quantity,

$$l = (a^2 + s^2)^{\frac{1}{2}} = a\left(1 + \frac{1}{2}\frac{s^2}{a^2}\right), \text{ nearly;}$$

hence, from (2),

$$a\left(1 + \frac{1}{2}\frac{s^2}{a^2}\right) = a(1 + \epsilon T), \qquad T = \frac{s^2}{2\epsilon a^3};$$

and therefore, by (1),

$$m \frac{d^2 s}{dt^2} = -\frac{s^3}{2\epsilon a^3 l} = -\frac{s^3}{2\epsilon a^4}, \text{ nearly;}$$

multiplying by $2\dfrac{ds}{dt}$, and integrating, we get

$$m\frac{ds^2}{dt^2} = C - \frac{s^4}{4\epsilon a^3};$$

let c be the value of s when $\dfrac{ds}{dt} = 0$; then

$$0 = C - \frac{c^4}{4\epsilon a^3},$$

and therefore

$$m\frac{ds^2}{dt^2} = \frac{1}{4\epsilon a^3}(c^4 - s^4);$$

hence, supposing s to be diminishing as t increases,

$$2a(\epsilon a m)^{\frac{1}{2}} \frac{ds}{(c^4 - s^4)} = -dt:$$

put $s = c\cos\phi$, and our equation becomes

$$\frac{2a}{c}(\epsilon a m)^{\frac{1}{2}} \frac{d\phi}{(1+\cos^2\phi)^{\frac{1}{2}}} = dt,$$

and therefore, 0, π, being the values of ϕ corresponding to the values c, $-c$, of s, the time of a complete oscillation will be equal to

$$\frac{2^{\frac{1}{2}}a}{c}(\epsilon a m)^{\frac{1}{2}} \int_0^\pi \frac{d\phi}{(1 - \frac{1}{2}\sin^2\phi)^{\frac{1}{2}}},$$

an elliptic function of which the modulus is $\sin\frac{1}{4}\pi$.

(6) A particle moves from rest from a distance a along a thin spiral tube towards a centre of force in the pole attracting inversely as the square of the distance; to find the whole time which will elapse before the particle will arrive at the centre of force, the equation to the spiral being

$$\log\frac{c}{r} = \frac{\theta}{\alpha}.$$

Let μ denote the absolute force; then, by (D), we have, for the velocity at any time,

$$v^2 = C - 2\int\frac{\mu}{r^2}\,dr = C + \frac{2\mu}{r},$$

but $v = 0$ when $r = a$; hence

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right),$$

or
$$\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2} = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right).$$

But, from the equation to the spiral,

$$d\theta = -\alpha\frac{dr}{r};$$

hence
$$\frac{dr^2}{dt^2}(1 + \alpha^2) = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right);$$

extracting the square root, and taking the negative sign because r decreases with the increase of t, we get

$$(1 + \alpha^2)^{\frac{1}{2}}\frac{dr}{\left(\frac{1}{r} - \frac{1}{a}\right)^{\frac{1}{2}}} = -2^{\frac{1}{2}}\mu^{\frac{1}{2}}dt,$$

whence
$$(1 + \alpha^2)^{\frac{1}{2}}\int\frac{dr}{\left(\frac{1}{r} - \frac{1}{a}\right)^{\frac{1}{2}}} = C - 2^{\frac{1}{2}}\mu^{\frac{1}{2}}t.$$

But
$$\int\frac{dr}{\left(\frac{1}{r} - \frac{1}{a}\right)^{\frac{1}{2}}} = a^{\frac{1}{2}}\int\frac{rdr}{(ar - r^2)^{\frac{1}{2}}}$$

$$= -a^{\frac{1}{2}}\int\frac{(\frac{1}{2}a - r)\,dr}{(ar - r^2)^{\frac{1}{2}}} + \tfrac{1}{2}a^{\frac{3}{2}}\int\frac{dr}{(ar - r^2)^{\frac{1}{2}}}$$

$$= -a^{\frac{1}{2}}(ar - r^2)^{\frac{1}{2}} + \tfrac{1}{2}a^{\frac{3}{2}}\operatorname{vers}^{-1}\frac{2r}{a}.$$

Hence we have

$$a^{\frac{1}{2}}(1 + \alpha^2)^{\frac{1}{2}}\left\{\tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2r}{a} - (ar - r^2)^{\frac{1}{2}}\right\} = C - 2^{\frac{1}{2}}\mu^{\frac{1}{2}}t.$$

Now $t = 0$, $r = a$, simultaneously; hence

$$\tfrac{1}{2}\pi a^{\frac{3}{2}}(1 + \alpha^2)^{\frac{1}{2}} = C;$$

also, T denoting the whole time of the approach to the pole,

$$0 = C - 2^{\frac{1}{2}}\mu^{\frac{1}{2}}T;$$

hence we have $$\tfrac{1}{2}\pi a^{\frac{3}{2}}(1+\alpha^2)^{\frac{1}{2}} = 2^{\frac{1}{2}}\mu^{\frac{1}{2}}T,$$

$$T = \frac{\pi a^{\frac{3}{2}}(1+\alpha^2)^{\frac{1}{2}}}{2^{\frac{3}{2}}\mu^{\frac{1}{2}}}.$$

(7) A particle descends by the action of gravity down a tube AO (fig. 137) in the form of a semi-cubical parabola of which the axis Ox is vertical, and the cusp the lowest point; to investigate the time of falling from a given point A to the cusp O.

Let $OM = x$, $PM = y$; then, by (B), since $X = -g$, $Y = 0$,

$$\frac{ds^2}{dt^2} = -2gx + C.$$

Let h be the initial value of OM, then, $\frac{ds}{dt}$ being initially zero, we have $$0 = -2gh + C,$$

and therefore $$\frac{ds^2}{dt^2} = 2g(h-x)\text{..............................}(1).$$

Now the equation to the curve is $ay^2 = x^3$,

and therefore $$a^{\frac{1}{2}}y = x^{\frac{3}{2}}, \quad a^{\frac{1}{2}}dy = \tfrac{3}{2}x^{\frac{1}{2}}dx,$$

$$ady^2 = \tfrac{9}{4}xdx^2, \quad ads^2 = \tfrac{1}{4}(9x+4a)\,dx^2.$$

Hence, by (1), there is

$$\frac{9x+4a}{4a}\frac{dx^2}{dt^2} = 2g(h-x),$$

and therefore $$dt = -\frac{1}{2(2ag)^{\frac{1}{2}}}\left(\frac{9x+4a}{h-x}\right)^{\frac{1}{2}}dx,$$

the negative sign being taken because x decreases as t increases.

Assume $z^2 = 9x + 4a$, and our equation becomes

$$dt = -\frac{1}{2(2ag)^{\frac{1}{2}}}\frac{z}{\left(h+\frac{4a}{9}-\frac{z^2}{9}\right)^{\frac{1}{2}}}\cdot\frac{2}{9}zdz$$

$$= -\frac{1}{3(2ag)^{\frac{1}{2}}}\frac{z^2dz}{(4a+9h-z^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{3(2ag)^{\frac{1}{2}}}\frac{z^2dz}{(\beta^2-z^2)^{\frac{1}{2}}}, \text{ where } \beta^2 = 4a+9h,$$

$$t = C - \frac{1}{3\,(2ag)^{\frac{1}{2}}} \int \frac{z^2 dz}{(\beta^2 - z^2)^{\frac{1}{2}}}.$$

But
$$\int \frac{z^2 dz}{(\beta^2 - z^2)^{\frac{1}{2}}} = -z\,(\beta^2 - z^2)^{\frac{1}{2}} + \int (\beta^2 - z^2)^{\frac{1}{2}}\, dz$$
$$= -z\,(\beta^2 - z^2)^{\frac{1}{2}} + \beta^2 \int \frac{dz}{(\beta^2 - z^2)^{\frac{1}{2}}} - \int \frac{z^2 dz}{(\beta^2 - z^2)^{\frac{1}{2}}}$$
$$= -\frac{1}{2} z\,(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \sin^{-1}\frac{z}{\beta};$$

hence
$$t = C - \frac{1}{3\,(2ag)^{\frac{1}{2}}} \left\{ -\frac{1}{2} z\,(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \sin^{-1}\frac{z}{\beta} \right\};$$

but, initially, $x = h$, $z^2 = \beta^2$, and therefore

$$0 = C - \frac{1}{3\,(2ag)^{\frac{1}{2}}} \cdot \frac{1}{2}\beta^2 \frac{\pi}{2}:$$

hence, eliminating C, we have

$$t = \frac{1}{3\,(2ag)^{\frac{1}{2}}} \left\{ \frac{1}{2} z\,(\beta^2 - z^2)^{\frac{1}{2}} + \frac{1}{2}\beta^2 \cos^{-1}\frac{z}{\beta} \right\},$$

and, substituting for z and β their values,

$$t = \frac{1}{3\,(2ag)^{\frac{1}{2}}} \left\{ \frac{3}{2}\,(9x + 4a)^{\frac{1}{2}}\,(h - x)^{\frac{1}{2}} + \frac{1}{2}\,(9h + 4a) \cos^{-1}\left(\frac{9x + 4a}{9h + 4a}\right)^{\frac{1}{2}} \right\}.$$

When the particle arrives at the cusp, $x = 0$, and therefore the whole time of descent is equal to

$$\frac{1}{3\,(2ag)^{\frac{1}{2}}} \left\{ 3a^{\frac{1}{2}} h^{\frac{1}{2}} + \frac{1}{2}\,(9h + 4a) \cos^{-1}\frac{2a^{\frac{1}{2}}}{(9h + 4a)^{\frac{1}{2}}} \right\}.$$

(8) Two particles P, P', (fig. 138), of which the masses are m, m', are connected by a straight rigid rod without weight, and, being constrained to move in two straight grooves Aa, Aa', which are inclined to the horizon at given angles and are in the same vertical plane, make small oscillations; to find the length of the isochronous pendulum.

Let AP, AP', make angles α, α', with the vertical line through A, and let $\angle PAP'$ be equal to ι. Let T denote the

mutual action of the two particles communicated along the rod. $AP = x$, $AP' = x'$, $PP' = c$, $\angle P'Pa = \phi$, $< PP'a' = \phi'$.

Then, for the motion of the two particles, we have, resolving forces along AP and AP',

$$m\frac{d^2x}{dt^2} = mg\cos\alpha - T\cos\phi,$$

$$m'\frac{d^2x'}{dt^2} = m'g\cos\alpha' - T\cos\phi'.$$

Eliminating T, we get

$$m\cos\phi'\frac{d^2x}{dt^2} - m'\cos\phi\frac{d^2x'}{dt^2} = mg\cos\alpha\cos\phi' - m'g\cos\alpha'\cos\phi.$$

But from the geometry we evidently have

$$c\cos\phi = x'\cos\iota - x, \qquad c\cos\phi' = x\cos\iota - x';$$

hence $$-m(x' - x\cos\iota)\frac{d^2x}{dt^2} + m'(x - x'\cos\iota)\frac{d^2x'}{dt^2}$$

$$= -mg\cos\alpha(x' - x\cos\iota) + m'g\cos\alpha'(x - x'\cos\iota) \text{........ (1).}$$

Let a, a', be the values of x, x', when there is equilibrium; then, $\frac{d^2x}{dt^2}$ and $\frac{d^2x'}{dt^2}$ being both equal to zero, we have

$$0 = -m\cos\alpha(a' - a\cos\iota) + m'\cos\alpha'(a - a'\cos\iota) \text{.........(2).}$$

Assume $x = a + v$, $x' = a' + v'$, v and v' being by the hypothesis small quantities. Then, from the equation (1), as far as small quantities of the first order, we have, by the aid of (2),

$$-m(a' - a\cos\iota)\frac{d^2v}{dt^2} + m'(a - a'\cos\iota)\frac{d^2v'}{dt^2}$$

$$= -mg\cos\alpha(v' - v\cos\iota) + m'g\cos\alpha'(v - v'\cos\iota) \text{....... (3).}$$

Now, by the geometry,

$$c^2 = x^2 + x'^2 - 2xx'\cos\iota;$$

hence $$0 = x\delta x + x'\delta x' - (x\delta x' + x'\delta x)\cos\iota.$$

But $\delta x = v$, $\delta x' = v'$, $x = a + v$, $x' = a' + v'$; hence, neglecting small quantities higher than of the first order,

$$0 = (a - a'\cos\iota)v + (a' - a\cos\iota)v' \text{.........(4).}$$

Let r represent the length of the isochronous pendulum; then

$$v = k \sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}} t + \epsilon\right\}, \qquad v' = k' \sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}} t + \epsilon\right\},$$

where k, k', ϵ, are constants: substituting these values of v, v', in the equations (3) and (4), we have

$$m\,(a' - a\cos\iota)\,\frac{k}{r} - m'\,(a - a'\cos\iota)\,\frac{k'}{r}$$

$$= k\,(m\cos\alpha\cos\iota + m'\cos\alpha') - k'\,(m'\cos\alpha'\cos\iota + m\cos\alpha),$$

and $$(a - a'\cos\iota)\,k + (a' - a\cos\iota)\,k' = 0.$$

Eliminating k and k' between these two equations,

$$\frac{m}{r}\,(a' - a\cos\iota)^2 + \frac{m'}{r}\,(a - a'\cos\iota)^2 = ma\cos\alpha\sin^2\iota + m'a'\cos\alpha'\sin^2\iota;$$

and therefore

$$r = \frac{m\,(a' - a\cos\iota)^2 + m'\,(a - a'\cos\iota)^2}{(ma\cos\alpha + m'a'\cos\alpha')\sin^2\iota} \quad \text{...........(5)}.$$

From P draw PO at right angles to AP, meeting $P'O$ drawn from P' at right angles to AP'. Then the projection of OP upon the line AP' is equal to $OP\sin\iota$, and the projection of PP' on the line AP' is equal to $a' - a\cos\iota$; but these two projections are evidently coincident; hence

$$OP^2\sin^2\iota = (a' - a\cos\iota)^2;$$

similarly $$OP'^2\sin^2\iota = (a - a'\cos\iota)^2.$$

Again, let G be the centre of gravity of m and m' in their position of equilibrium, and H the point in which a vertical line through G will cut a horizontal line through A; then we have

$$(m + m')\,GH = ma\cos\alpha + m'a'\cos\alpha'.$$

Hence, from (5), we obtain

$$r = \frac{m\,.\,OP^2 + m'\,.\,OP'^2}{(m + m')\,GH}.$$

(9) A particle slides down a plane of given length, inclined at an angle θ to the horizon, and is reflected by the horizontal plane; to determine the value of θ that the range on the

horizontal plane may be the greatest possible, the particle being perfectly elastic.

$$\theta = \sin^{-1}\{(\tfrac{2}{3})^{\frac{1}{2}}\}.$$

(10) An ellipse is placed with its major axis vertical; to find the radius vector by which a particle will descend in the shortest time from the upper focus to the curve.

If $\theta =$ the inclination of the required radius vector to the vertical line drawn downwards from the focus; then $\theta = 0$, if $e < \frac{1}{2}$; and $\theta = \cos^{-1}\left(\frac{1}{2e}\right)$, if $e > \frac{1}{2}$.

(11) An ellipse is placed with its plane inclined to a horizontal plane and with its major axis in that plane: to determine the position of the line drawn from the focus to the perimeter of the ellipse, down which a particle acted on by gravity will descend in the shortest time.

If θ denote the inclination of the required line to the distance of the focus from the more remote apse,

$$\cos\theta = \frac{1 - (8e^2 + 1)^{\frac{1}{2}}}{4e}.$$

(12) Two equal spherical particles of given elasticity are placed at two points in the circumference of a vertical circle, the radii of these two points making angles of 60° on each side of the radius which tends vertically downwards; to determine the sum of the chords of the arcs described by each particle before it ceases to move.

If $a =$ the radius of the circle, and $e =$ the common elasticity of the particles, the required space will be equal to

$$\frac{a}{1 - e}.$$

(13) A particle is placed within a thin rectilinear tube, and is attracted by a force always tending towards a fixed point without the tube, and varying as some function of the distance; to find the time of a small oscillation of the particle.

If $f(r)$ denote the intensity of the force at a distance r, and a be the perpendicular distance of the centre of force from the tube, the time of a small oscillation will be equal to

$$\frac{\pi a^{\frac{1}{2}}}{\{f(a)\}^{\frac{1}{2}}}.$$

(14) Two equal particles, attracting each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines at right angles to each other; supposing their motions to commence from rest, to find the time in which each of them will arrive at the intersection of the two straight lines.

If a denote the initial distance between the particles, and μ the absolute attracting force of each, they will arrive simultaneously at the intersection of the straight lines in a time equal to

$$\frac{\pi a^{\frac{3}{2}}}{2\,(2\mu)^{\frac{1}{2}}}.$$

The particles would arrive simultaneously at the intersection of their paths for any other law of mutual attraction.

(15) A particle acted on by gravity descends from any point in the arc of an inverted cycloid, of which the axis is vertical, to the lowest point of the curve; to find the whole time of descent.

If a be the radius of the generating circle, the required time will be equal to

$$\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

This result, being independent of the initial position of the particle, shews that the time of descent will be the same from whatever point in the curve the motion commences. This elegant mechanical property of the Cycloid, from which it has received the name of a Tautochronous Curve, was first discovered by Huyghens, *Horolog. Oscill.* Pars II.

(16) A particle falls from rest towards a fixed centre of force, which attracts directly as the distance: to find the equation to the path of the particle, supposing it to be included in a thin smooth rectilinear tube revolving in one plane with a uniform angular velocity and passing through the centre of force.

Let $\mu =$ the absolute force, $\omega =$ the angular velocity of the tube; and let a, the initial distance of the particle from the centre of force, be taken as the prime radius vector: then the equation to the path will be

$$r = \tfrac{1}{2}\, a \left\{e^{(\omega^2-\mu)^{\frac{1}{2}}\frac{\theta}{\omega}} + e^{-(\omega^2-\mu)^{\frac{1}{2}}\frac{\theta}{\omega}}\right\},$$

or

$$r = a \cos\left\{(\mu - \omega^2)^{\frac{1}{2}}\,\frac{\theta}{\omega}\right\},$$

accordingly as μ is less or greater than ω^2.

If $\mu = \omega^2$, the path becomes a circle.

(17) A particle, attracted by a force always tending towards a point A, (fig. 139), and varying directly as the distance, describes the arc OP and the chord OP of a fixed smooth curve in the same time, whatever point P be chosen in the curve: the particle has no motion when at O. To find the nature of the curve.

The curve is the Lemniscata, the centre of which is at O and of which the axis is inclined to OA at an angle $\dfrac{\pi}{4}$.

Bonnet; *Liouville, Journal de Mathematiques*, xv., 1844.

(18) A particle under the action of gravity falls down an arc OB (fig. 140) of one of the loops of a Lemniscata, of which the axis OA is inclined at an angle of 45° to the horizon; to determine the time of the descent.

Let a denote the semi-axis of the corresponding equilateral hyperbola, θ the angle between the chord OB and the axis OA of the loop, and T the required time. Then

$$T = \left(\frac{8^{\frac{1}{2}}a}{g}\right)^{\frac{1}{2}} . \{\tan(\tfrac{1}{4}\pi - \theta)\}^{\frac{1}{2}}.$$

This expression for the time is the same as that for the descent of a particle down the chord OB; a mechanical property of the Lemniscata which was discovered by Saladini, *Memorie dell' Istituto Nazionale Italiano*, Tom. I. parte 2.

(19) A spherical particle A impinges with a velocity u in a horizontal direction upon a spherical particle B, which is resting at the lowest point of an inverted cycloid with its axis vertical; to determine the velocities of A and B after any number of impacts, the volumes of the particles being equal, while their masses differ in any proposed degree.

The velocities of A, B, after x impacts, will be respectively, e denoting their common elasticity,

$$\frac{A - A\,(-e)^x}{A+B}\,u, \qquad \frac{A + B\,(-e)^x}{A+B}\,u.$$

SECT. 2. *Pressure of a moving Particle on immoveable plane Curves.*

The general value of the reaction of a curve against a particle which is moving along the curve, is given by the formula

$$R = Y\frac{dx}{ds} - X\frac{dy}{ds} \pm \frac{1}{\rho}\frac{ds^2}{dt^2} = N \pm \frac{v^2}{\rho}\;\ldots\ldots\ldots\ldots(\mathrm{A}),$$

where N represents the resolved part of the whole accelerating force on the particle estimated along the normal in an opposite direction to that in which the reaction R exerts itself, and ρ denotes the radius of curvature of the curve. In this formula the positive or the negative sign is to be taken accordingly as the particle is moving on the concave or on the convex side of the curve.

This formula was first given by L'Hôpital[1] in the discussion of John Bernoulli's problem of the Curve of Equal Pressure.

When the expression for R becomes equal to zero, the particle

[1] *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 9.

will either leave the curve or will move along it freely without experiencing any reaction; and the analytical condition

$$\frac{v^2}{\rho} = \mp N$$

shews that, on the commencement of free motion, the normal accelerating force and the centrifugal force of the particle must be equal and opposite.

(1) A particle, starting with a given velocity from the vertex of a parabola, of which the axis is vertical, descends down the convex side of the curve by the action of gravity; to find the reaction of the curve at any point of the descent.

The resolved part of the force of gravity along the normal in a direction opposite to the reaction is $g\dfrac{dy}{ds}$, and therefore by (A), the particle moving on the convex side of the curve,

$$R = g\frac{dy}{ds} - \frac{1}{\rho}\frac{ds^2}{dt^2}.$$

Now, the equation to the parabola being $y^2 = 4mx$,

$$\frac{dy}{ds} = \frac{m^{\frac{1}{2}}}{(m+x)^{\frac{1}{2}}} \text{ and } \rho = \frac{2}{m^{\frac{1}{2}}}(m+x)^{\frac{3}{2}}.$$

Also, if h be the altitude due to the initial velocity of the particle, we have

$$\frac{ds^2}{dt^2} = 2g(x+h).$$

Hence
$$R = \frac{gm^{\frac{1}{2}}}{(m+x)^{\frac{1}{2}}} - \frac{gm^{\frac{1}{2}}}{(m+x)^{\frac{3}{2}}}(x+h)$$

$$= m^{\frac{1}{2}}g\frac{m-h}{(m+x)^{\frac{3}{2}}}.$$

If $h = m$, then the pressure during the whole motion will be equal to zero; and the particle will describe the parabola freely. If h were greater than m, since, from the nature of the case, R cannot have any negative value, the particle would

from the first proceed in a path different from the parabola in question. If, instead of supposing the particle to move on a mere curve, we were to conceive it to be moving within an indefinitely thin parabolic tube, R might be negative; and in fact always would be negative, supposing h to be greater than m, when the motion would be the same as if the particle were moving along the concave side of the parabolic curve.

Euler; *Mechan.* Tom. II. p. 64.

(2) A particle, starting from rest, descends down the convex side of a circle from a given point in its circumference; to find where it will leave the curve.

Let O (fig. 141) be the centre of the circle, AO being a vertical radius. Let P be the initial position of the particle, Q its point of departure; PM, QN, horizontal lines. Join OQ, and let $\angle AOQ = \phi$; $a =$ the radius of the circle.

Then, the centrifugal force at Q being equal to the normal component of gravity, we have

$$\frac{v^2}{a} = g \cos\phi;$$

but, denoting MN by x,

$$v^2 = 2gx;$$

hence, putting $MO = c$,

$$2x = a\cos\phi = c - x, \quad x = \tfrac{1}{3}c.$$

Fontana; *Memorie della Societa Italiana*, 1782, p. 175.

(3) A particle is moving along the convex side of an equiangular spiral, towards the pole of which it is attracted by a force varying as any power of the distance; to determine the reaction of the curve at any time during the motion.

Let r be the distance of the particle from the pole at any time, μr^n the attractive force, α the constant angle between the curve and the radius vector, β the initial velocity, and a the initial value of r. Then, by the formula (A), N being equal to $\mu r^n \sin \alpha$, we have

$$R = \mu r^n \sin\alpha - \frac{v^2}{\rho} \quad\text{.......................... (1).}$$

Again, estimating the velocity v of the particle in a direction corresponding to an increase of r, and denoting by ds an element of its path, we have

$$v\frac{dv}{ds} = -\mu r^n \cos\alpha \dots\dots\dots\dots\dots\dots (2).$$

Now, by the nature of the curve, $\cos\alpha\, ds = dr$; hence, from (2),

$$v\frac{dv}{dr} = -\mu r^n;$$

integrating and observing that β, a, are the initial values of v, r, we get

$$v^2 - \beta^2 = -\frac{2\mu}{n+1}(r^{n+1} - a^{n+1}) \dots\dots\dots\dots (3).$$

Again, p denoting the perpendicular from the pole upon the tangent to the curve, we have, since $p = r\sin\alpha$,

$$\rho = r\frac{dr}{dp} = \frac{r}{\sin\alpha} \dots\dots\dots\dots\dots\dots\dots\dots\dots (4).$$

From (1), (3), (4), we obtain

$$R = \mu r^n \sin\alpha - \frac{\sin\alpha}{r}\left\{\beta^2 - \frac{2\mu}{n+1}(r^{n+1} - a^{n+1})\right\}$$

$$= \mu\frac{n+3}{n+1}r^n \sin\alpha - \frac{\beta^2\sin\alpha}{r} - \frac{2\mu\sin\alpha}{(n+1)r}a^{n+1}.$$

Euler; *Mechan.* Tom. II. p. 86.

(4) A particle attracted towards two centres of force, varying inversely as the square of the distance, moves in an hyperbolic groove, of which the foci are the centres of force; to find the pressure on the groove at any point, the particle being supposed to move on the concave side.

Let P (fig. 142) be the position of the particle at any time t; S, H, the foci of the hyperbola; let $SP = r$, $HP = r'$; let a be the transverse semi-axis; PT a tangent at P; let $\angle SPT = \phi = \angle HPT$; let μ, μ', be the absolute forces towards S, H.

Resolving forces at right angles to the tangent at P, we have, by the equation (A),

$$R = \left(\frac{\mu'}{r'^2} - \frac{\mu}{r^2}\right)\sin\phi + \frac{v^2}{\rho} \dots\dots\dots\dots\dots\dots (1);$$

also, for the value of v at any time, there is

$$v^2 = 2\int\left(-\frac{\mu'}{r'^2}\,dr' - \frac{\mu}{r^2}\,dr\right) + C$$?

$$= \frac{2\mu'}{r'} + \frac{2\mu}{r} + C.$$

Let f, f', be the initial values of r, r', and β the initial value of v; then

$$\beta^2 = \frac{2\mu'}{f'} + \frac{2\mu}{f} + C,$$

and therefore $$v^2 = \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2.$$

Hence, from (1), we have

$$R\rho = \left(\frac{\mu'}{r'^2} - \frac{\mu}{r^2}\right)\rho\sin\phi + \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2.$$

But $2\rho\sin\phi$ is equal to the chord of curvature through S, which, by the nature of the hyperbola, is equal to $\frac{2rr'}{a}$; hence

$$R\rho = \left(\frac{\mu'}{r'^2} - \frac{\mu}{r^2}\right)\frac{rr'}{a} + \frac{2\mu'}{r'} + \frac{2\mu}{r} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'(r+2a)}{ar'} - \frac{\mu(r'-2a)}{ar} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'}{a} - \frac{\mu}{a} - \frac{2\mu'}{f'} - \frac{2\mu}{f} + \beta^2$$

$$= \frac{\mu'(f'-2a)}{af'} - \frac{\mu(f+2a)}{af} + \beta^2$$

$$= \frac{\mu' f}{af'} - \frac{\mu f'}{af} + \beta^2.$$

If the initial velocity be zero, and the particle be attracted at the commencement of its motion with equal intensity by the two centres of force; then $\beta = 0$, $\frac{\mu}{f^2} = \frac{\mu'}{f'^2}$, and therefore $R = 0$ during the whole motion. Hence the particle would under these circumstances describe the hyperbola freely.

6 (5) A particle, acted on by gravity, oscillates in a circular arc; to find the reaction of the curve at any point.

Let O (fig. 143) be the centre of the circle; P the position of the particle at any time; A the lowest point of the circle; let $\angle AOP = \theta$. Then, θ being initially equal to α and the velocity zero, we have

$$R = g\,(3 \cos \theta - 2 \cos \alpha).$$

(6) A series of circles in a vertical plane have a common highest point; a particle, starting at this point, slides down the convex side of each circle: to find the locus of the points where the particles leave the circles.

The required locus is a straight line, passing through the highest point of the circles, and making an angle $\tan^{-1}(\sqrt{5})$ with the vertical.

(7) A particle is projected with a given velocity at the highest point of a circle in a vertical plane along the concave side of the curve; to determine the pressure on the curve at any point in its path.

Let AOB (fig. 144) be the vertical diameter, O being the centre of the circle; P the position of the particle at any time; let $OP = a$, $\angle AOP = \theta$; let β be the velocity of projection at A; then, for the pressure at P,

$$R = \frac{\beta^2}{a} + g\,(2 - 3 \cos \theta).$$

Suppose that $R = 0$ initially; then $\dfrac{\beta^2}{a} = g$, and

$$\begin{aligned} R &= 3g \text{ vers } \theta \\ &= 6g, \qquad \text{when } \theta = \pi; \end{aligned}$$

which shews that, when the particle arrives at the lowest point, the reaction is six times the force of gravity.

Euler; *Mechan.* Tom. II. p. 65, Cor. 7.

(8) A particle descends down the convex side of a logarithmic curve placed with its asymptote parallel to the horizon; to find where it leaves the curve.

Let P (fig. 145) be the point at which the particle is placed, and Q the point of its departure; let $OM = h$, $ON = x$. Then, the

equation to the curve being $y = \log_a x$, we have, putting $A = \log_e a$,

$$x = \frac{1}{A}\left\{Ah + (1 + A^2h^2)^{\frac{1}{2}}\right\}.$$

Fontana; *Memorie della Societa Italiana*, 1782, p. 182.

(9) A particle descends from rest down the convex side of an ellipse with its major axis vertical, from a given point in the curve; to determine where it will leave the ellipse.

Let the highest point of the ellipse be taken as the origin of co-ordinates, the axis of x being vertical, and that of y horizontal. Let a, b, denote the semi-axes major and minor; h the initial distance of the particle from the axis of y, and x the distance of the point at which it leaves the curve. Then the value of x will be a root of the cubic equation

$$(a^2 - b^2)(x^3 - 3ax^2) - 3a^2b^2x + a^3(b^2 + 2ah) = 0.$$

Fontana; *Ib.* p. 175.

(10) A particle descends from rest down the convex side of the Cissoid of Diocles, which is so placed as to have its asymptote vertical; the initial place of the particle being known, to find the point in which it will leave the curve.

Let P (fig. 146) be the initial position of the particle, and Q its place on leaving the curve: draw PS, QN, at right angles to Ox, Oy; let $PS = h$, $QN = x$. Then, a being the radius of the generating circle, the value of x will be a root of the cubic equation

$$x^3 - \frac{16a}{9}x^2 + \frac{64a^2 + 36h^2}{81}x - \frac{8ah^2}{9} = 0.$$

If the motion commence at the cusp O, $h = 0$, and therefore

$$x = \frac{8}{9}a.$$

Fontana; *Ib.* p. 181.

(11) A particle is projected with a given velocity along the convex side of a parabola from a given point in the curve; to determine the reaction of the curve at any time of the

motion, the particle being always attracted to the focus by a force varying inversely as the square of the distance.

Let S (fig. 147) be the focus of the parabola; B the point from which the particle is initially projected in the direction of the tangent BT; P the position of the particle after any time; let $SP=r$, $SB=a$, $SA=m$, $\beta=$ the velocity of projection, $\mu=$ the absolute force towards S. Then, at P,

$$R=\left(\frac{m}{r^3}\right)^{\frac{1}{2}}\left(\frac{\mu}{a}-\frac{\beta^2}{2}\right).$$

(12) There is a centre of force at one extremity of the diameter of a semi-circle, the force being repulsive and varying as the distance: to find the pressure exerted upon the curve by a body which moves from rest from the centre of force along its concave side, and the time which elapses before it reaches the other extremity of the diameter.

If $a=$ the radius of the circle, $m=$ the mass of the body, $\mu=$ the absolute force, and $R=$ the pressure when the body is at a distance r from the centre of force,

$$R=\frac{3\mu m r^2}{2a};$$

and the required time is infinite.

(13) A particle is attached to the end of a fine thread which just winds round the circumference of a circle, in the centre of which there is a repulsive force varying as the distance: to find the time of unwinding, and the tension of the string at any time.

If $\mu=$ the absolute force, and $a=$ the radius of the circle, the time of unwinding is equal to $\dfrac{2\pi}{\sqrt{\mu}}$, and the tension at any time t is equal to $2\mu^{\frac{3}{2}}\,.\,a\,.\,t$.

(14) An ellipse is placed with its major axis in a vertical position; to find the velocity with which a particle must be projected vertically upwards from the extremity of the minor axis

along the interior of the elliptic arc, so that after quitting the curve it may pass through the centre.

If a, b, denote the semi-axes major and minor, the required velocity will be equal to

$$\left\{\frac{(8a^2+b^2)\,g}{3a\,.\,3^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

(15) A particle moves along the convex side of an ellipse under the action of two forces tending to the foci and varying inversely as the square of the distance, and a third force tending to the centre and varying as the distance; to find the reaction of the curve at any point.

Let R denote the reaction of the curve on the particle at any point, ρ the radius of curvature; f, f', the initial focal distances, and μ, μ', the corresponding absolute forces; μ'' the absolute force to the centre, $2a$ the axis major of the ellipse, and β the initial velocity. Then

$$R\rho = \frac{\mu f'}{af} + \frac{\mu' f}{af'} + \mu'' ff' - \beta^2.$$

If β', β'', β''', denote the velocities which the particle ought to have initially to revolve freely round the three centres of force taken separately,

$$\beta'^2 = \frac{\mu f'}{af}, \quad \beta''^2 = \frac{\mu' f}{af'}, \quad \beta'''^2 = \mu'' ff';$$

and therefore, when the forces are taken conjointly, it will revolve about them freely when

$$\beta^2 = \beta'^2 + \beta''^2 + \beta'''^2.$$

SECT. 3. *Inverse Problems on the Motion of a Particle along immoveable plane Curves.*

(1) To find a curve EPF (fig. 148) such that, A and B being two given points in the same horizontal line, the sum of the times in which a particle will descend by the action of

gravity down the straight lines AP, BP, may be the same whatever point in the curve P may be.

Bisect AB in O; let Ox, a vertical line, be the axis of x, and OAy, which is horizontal, the axis of y; let $AB = 2a$. Then, x, y, being the co-ordinates of P, the times down AP, BP, will be respectively equal to

$$\left\{\frac{x^2 + (a-y)^2}{\frac{1}{2}gx}\right\}^{\frac{1}{2}}, \quad \left\{\frac{x^2 + (a+y)^2}{\frac{1}{2}gx}\right\}^{\frac{1}{2}}.$$

Hence, k denoting the sum of the times,

$$(\tfrac{1}{2}k^2gx)^{\frac{1}{2}} = \{x^2 + (a-y)^2\}^{\frac{1}{2}} + \{x^2 + (a+y)^2\}^{\frac{1}{2}} \dots\dots\dots\dots\dots \text{(1)}.$$

Putting $\frac{1}{2}k^2g = 4c$, and squaring both sides of the equation, we have

$$2cx = a^2 + x^2 + y^2 + \{x^2 + (a-y)^2\}^{\frac{1}{2}}\{x^2 + (a+y)^2\}^{\frac{1}{2}},$$

$$(2cx - a^2 - x^2 - y^2)^2 = \{x^2 + (a-y)^2\}\{x^2 + (a+y)^2\}.$$

Developing both sides of the equation, and simplifying, we shall readily find that

$$c^2x^2 - a^2cx - cx^3 - cxy^2 = -a^2y^2,$$

and therefore

$$y^2 = cx\,\frac{x^2 - cx + a^2}{a^2 - cx} \dots\dots\dots\dots\dots\dots\dots \text{(2)},$$

which is the equation to the required curve.

If we trace this curve, we shall find it to consist of a branch VOV' having an asymptote parallel to the axis of y, and of an oval EFP. The oval is the portion of the curve which corresponds to the problem which we are considering. The infinite branch VOV' would correspond to the condition that the times down AP', BP', shall have a constant difference: in which case we should have had, instead of the equation (1),

$$(\tfrac{1}{2}k^2gx)^{\frac{1}{2}} = \{x^2 + (a+y)^2\}^{\frac{1}{2}} - \{x^2 - (a-y)^2\}^{\frac{1}{2}};$$

whence, by the involution, we should have obtained the same equation (2). The curve has pretty much the shape of the Conchoid, although its equation is essentially different.

Fuss; *Mémoires de l'Acad. de St. Pétersb.* 1819.

(2) A particle, not acted on by any forces, is constrained to move within a thin tube of such a form that the acceleration of the particle parallel to a given straight line is invariable; to determine the equation to the path of the particle.

Let the axis of x be taken parallel to the given line; let c be the constant acceleration of the particle parallel to the axis of x, and β its velocity within the tube, which will be invariable. Then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \beta^2 \text{ .... } (1);$$

but, by the condition of the problem, $\dfrac{d^2x}{dt^2} = c$, and therefore, the axis of y being so chosen that $\dfrac{dx}{dt} = 0$ when $x = 0$,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2c\frac{dx}{dt}, \qquad \frac{dx^2}{dt^2} = 2cx \text{ .... } (2).$$

Eliminating dt between the equations (1) and (2), we get

$$2cx\left(1 + \frac{dy^2}{dx^2}\right) = \beta^2,$$

or, putting $\dfrac{\beta^2}{4c} = a$,

$$\frac{dy}{dx} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}};$$

whence, by integration, the position of the axis of x being supposed such that $x = 0$ when $y = 0$,

$$y = (2ax - x^2)^{\frac{1}{2}} + a \operatorname{vers}^{-1}\frac{x}{a};$$

which is the equation to a cycloid of which the axis is parallel to the given line.

There is an elaborate investigation by Euler, in the *Mémoires de l'Académie de St. Pétersb.*, Tom. x. p. 7, on the nature of the curve of constraint when the particle is subject to the action of gravity, and the direction of uniform acceleration is horizontal. A notice of this problem may be seen in the *Bulletin des Sciences de Bruxelles*, Tom. IX.

(3) To determine the curve down which a particle may descend by the action of gravity, so as to describe equal vertical spaces in equal times, the tangent to the curve at the point where the motion commences being vertical.

Let O (fig. 149) be the point where the motion commences, Ox the axis of x touching the required curve OA at O, Oy the axis of y at right angles to Ox; let $OM = x$, $PM = y$, $\beta =$ the invariable velocity of the particle parallel to Ox.

Then $\dfrac{ds^2}{dt^2} = C + 2gx$, C being a constant quantity,

$$\frac{ds^2}{dx^2}\frac{dx^2}{dt^2} = C + 2gx.$$

But $\dfrac{dx}{dt} = \beta$; hence

$$\beta^2 + \beta^2\frac{dy^2}{dx^2} = C + 2gx.$$

But, when $x = 0$, $\dfrac{dy}{dx} = 0$; and therefore

$$\beta^2 = C, \quad \beta^2\frac{dy^2}{dx^2} = 2gx,$$

$$\frac{dy}{dx} = \frac{(2g)^{\frac{1}{2}}}{\beta}x^{\frac{1}{2}}, \qquad y = \frac{2\,(2g)^{\frac{1}{2}}}{3\beta}x^{\frac{3}{2}},$$

no constant being added because $x = 0$, $y = 0$, simultaneously. The required curve OA is therefore the semi-cubical parabola, O being the cusp, and Ox the axis.

This curve is called the Isochrone. It was proposed by Leibnitz[1], as a challenge to the disciples of Des Cartes, who, from an excessive attachment to the geometry of their master, affected to despise the methods of the Differential Calculus. No solution was communicated by any of the Cartesians. Huyghens alone successfully accepted the challenge, by whom a geometrical solution was given in the *Nouvelles de la Republique des Lettres, Octobre* 1687. The solution by Leibnitz appeared for the first time in the *Acta Erudit. Lips.* 1689, p. 196 et sq. The solutions

[1] *Nouvelles de la Republique des Lettres, Septembre* 1687.

both of Huyghens and of Leibnitz were synthetical. An analytical solution was given afterwards for the first time by James Bernoulli[1].

(4) A particle is projected with a given velocity from a point A (fig. 150) along a horizontal line AO towards a point O; to find the curve along which it must be constrained to move that it may approach the point O uniformly; the particle being acted on by gravity, and AO being a tangent to the required curve.

Let P be the position of the particle at any time; let $AO = a$, $OP = r$, $\angle AOP = \theta$; $\beta =$ the velocity of the particle at its initial position A. For the motion of the particle at any point in its descent there is

$$\frac{dr^2}{dt^2} + r^2 \frac{d\theta^2}{dt^2} = \beta^2 + 2gr \sin\theta,$$

or

$$\frac{dr^2}{dt^2}\left(1 + r^2 \frac{d\theta^2}{dr^2}\right) = \beta^2 + 2gr \sin\theta.$$

But, by the condition of the problem, $\frac{dr}{dt} = C$, a constant quantity: hence

$$C^2\left(1 + r^2 \frac{d\theta^2}{dr^2}\right) = \beta^2 + 2gr \sin\theta.$$

But, initially, $\theta = 0$, $r\frac{d\theta}{dr} = 0$; hence $C^2 = \beta^2$, and therefore

$$\beta^2 r^2 \frac{d\theta^2}{dr^2} = 2gr \sin\theta,$$

$$\frac{dr}{r^{\frac{1}{2}}} = \frac{\beta}{(2g)^{\frac{1}{2}}} \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}}:$$

integrating, and observing that $\theta = 0$, $r = a$, initially; we have

$$r^{\frac{1}{2}} - a^{\frac{1}{2}} = \frac{\beta}{2(2g)^{\frac{1}{2}}} \int_0^\theta \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}},$$

which is an equation for the construction of the path of the particle.

[1] *Act. Erudit. Lips.* 1690, p. 217.

The particle will move from A along $ABCO$ to the point O with a uniform velocity of approach: it will afterwards move from O along $OcBa$ with a uniform velocity of recession. When it has arrived at a, it will proceed uniformly along Oa produced.

This curve has been called the Paracentric Isochrone by Leibnitz, by whom the problem was originally proposed as a challenge to the mathematicians of the day, in the *Acta Erudit. Lips.* 1689, p. 198. Several years elapsed before the problem received a solution. At length James Bernoulli succeeded in obtaining one, which appeared in the *Acta Erudit. Lips.* 1694, p. 277. Solutions were shortly afterwards published by Leibnitz and John Bernoulli, in the *Acta Erudit. Lips.* 1694, p. 371, 394. The problem was afterwards generalized by Varignon in the *Mémoires de l'Académie des Sciences de Paris*, 1699, p. 9 et sq.

(5) To find the nature of the curve OPA (fig. 151) such that a particle acted on by gravity will descend down any arc OP in the same time as down its chord.

Let Ox be vertical, Oy horizontal, PM parallel to yO. Let $OP=r$, $\angle xOP=\theta$, arc $OP=s$, $OM=r\cos\theta$. Then, since the velocity acquired down the arc OP is the same as that which is due to falling freely down OM,

$$\frac{ds^2}{dt^2}=2gr\cos\theta,\quad dt=\frac{1}{(2g)^{\frac{1}{2}}}\,\frac{ds}{(r\cos\theta)^{\frac{1}{2}}};$$

and therefore the whole time of descent down OP is equal to

$$\frac{1}{(2g)^{\frac{1}{2}}}\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}}.$$

But the time of descent down the chord OP is equal to

$$\left(\frac{2}{g}\right)^{\frac{1}{2}}\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}};$$

and, therefore, by hypothesis,

$$\left(\frac{2}{g}\right)^{\frac{1}{2}}\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}}=\frac{1}{(2g)^{\frac{1}{2}}}\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}},$$

$$2\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}}=\int_0^s\frac{ds}{(r\cos\theta)^{\frac{1}{2}}}.$$

Differentiating both sides of the equation, we have

$$\left(\frac{\cos\theta}{r}\right)^{\frac{1}{2}}\frac{\cos\theta\,dr+r\sin\theta d\theta}{(\cos\theta)^2}=\frac{ds}{(r\cos\theta)^{\frac{1}{2}}},$$

$$\cos\theta\,dr+r\sin\theta\,d\theta=\cos\theta\,(dr^2+r^2d\theta^2)^{\frac{1}{2}}.$$

Squaring both sides and simplifying,

$$2\sin\theta\cos\theta\,rdrd\theta=r^2\cos 2\theta\,d\theta^2,$$

$$\frac{dr}{r}=\frac{\cos 2\theta}{\sin 2\theta}d\theta.$$

Integrating,

$$\log r^2=\log a^2+\log\sin 2\theta,\quad r^2=a^2\sin 2\theta,$$

where a^2 is some constant quantity.

From O draw OE, bisecting the angle xOy, and let $\angle POE=\phi$; then, since $\theta=\frac{1}{4}\pi-\phi$, we have

$$r^2=a^2\cos 2\phi;$$

which is the equation to the Lemniscata of James Bernoulli, O being the centre and A the vertex of the equilateral hyperbola. This very beautiful problem is due to Saladini.

Saladini; *Memorie dell' Istituto Nazionale Italiano*, Tom. I. parte 2. Fuss; *Mémoires de l'Acad. de St. Pétersb.* 1819.

(6) To find the equation to the tautochrone when a particle is acted on by any forces whatever in one plane.

A tautochrone is a curve along which a particle acted on by any assigned forces will arrive in the same time at a given point from whatever point in the curve its motion commences. Let A (fig. 152) be any assigned point, and E any point whatever in the curve AEB; then the time from E to A is to be independent of the position of E.

Let P be any point in AE; let $AP=s$, $AE=a$, $S=$ the sum of the resolved parts of the accelerating forces on the particle along the tangent PT at the point P. Then, for the motion of the particle,

$$\frac{d^2s}{dt^2}=-S, \quad \frac{ds^2}{dt^2}=C-2\int Sds.$$

But, the particle being supposed to have no initial velocity,

$$0=C-2\int^a Sds;$$

and therefore

$$\frac{ds^2}{dt^2}=2\int_s^a Sds \dots\dots\dots\dots (1),$$

$$dt=-\frac{1}{2^{\frac{1}{2}}}\frac{ds}{\left(\int_s^a Sds\right)^{\frac{1}{2}}}.$$

The time from E to A is equal to

$$\frac{1}{2^{\frac{1}{2}}}\int_0^a\frac{ds}{\left(\int_s^a Sds\right)^{\frac{1}{2}}};$$

and this formula must be independent of a. Hence we must have

$$\int\frac{ds}{\left(\int_s^a Sds\right)^{\frac{1}{2}}}=\phi\left(\frac{s}{a}\right),$$

where $\phi\left(\frac{s}{a}\right)$ denotes some function of $\frac{s}{a}$. Hence

$$\frac{1}{\left(\int_s^a Sds\right)^{\frac{1}{2}}}=\frac{1}{a}\phi'\left(\frac{s}{a}\right), \quad \int_s^a Sds=\frac{a^2}{\left\{\phi'\left(\frac{s}{a}\right)\right\}^2};$$

and therefore, differentiating with respect to s,

$$-Sds=d\frac{a^2}{\left\{\phi'\left(\frac{s}{a}\right)\right\}^2}.$$

But, the tautochrone AB being an invariable curve, whatever be the value of a, it is manifest that a must not appear in this equation; hence

$$\left\{\phi'\left(\frac{s}{a}\right)\right\}^2=-\frac{a^2}{As^2}, \text{ where } A \text{ is a constant quantity,}$$

and therefore $$S = ks \quad\text{..............................}\quad (2),$$
k being some constant quantity.

Hence, by (1) and (2),

$$\frac{ds^2}{dt^2} = k\,(a^2 - s^2),$$

and therefore, if τ denote the time of the motion from E to A,

$$\tau = -\frac{1}{k^{\frac{1}{2}}}\int_a^0 \frac{ds}{(a^2 - s^2)^{\frac{1}{2}}} = \frac{\pi}{2k^{\frac{1}{2}}}, \qquad k = \frac{\pi^2}{4\tau^2}.$$

Hence we have, from (2),

$$S = \frac{\pi^2 s}{4\tau^2},$$

which is a differential equation to the tautochrone.

The direct problem of Tautochronism in the case when gravity is the accelerating force, was first considered by Huyghens, in his *Horolog. Oscill.*, where he proves the inverted cycloid with its axis vertical to be tautochronous. The inverse problem was first considered by Newton, *Princip.* Lib. I. sect. 10. See also Euler, *Comment. Petrop.* 1729, and *Mechan.* Tom. II. p. 211.

(7) A particle is acted on by an attractive force tending towards a fixed centre, and varying as the distance; to find the tautochrone.

Let μ denote the absolute force of attraction, r the radius vector at any point of the curve, p the perpendicular from the pole upon the tangent at the point, ϕ the inclination of the tangent to the radius vector.

Then, by the formula of the preceding general problem, we have, putting $\mu r \cos\phi$ for S,

$$\mu r \cos\phi = \frac{\pi^2 s}{4\tau^2},$$

whence $$\mu d\,(r \cos\phi) = \frac{\pi^2 ds}{4\tau^2};$$

but $ds \cos\phi = dr$; hence we have

$$\mu r \cos\phi\, d(r \cos\phi) = \frac{\pi^2}{4\tau^2}\, r dr;$$

integrating, we get

$$\mu r^2 \cos^2\phi + C = \frac{\pi^2 r^2}{4\tau^2},$$

or

$$\mu (r^2 - p^2) + C = \frac{\pi^2 r^2}{4\tau^2}.$$

Let c be the value of r when $s = 0$, and therefore when $\phi = \frac{1}{2}\pi$; then also $p = c$, and consequently

$$C = \frac{\pi^2 c^2}{4\tau^2}.$$

Hence

$$\mu (r^2 - p^2) + \frac{\pi^2 c^2}{4\tau^2} = \frac{\pi^2 r^2}{4\tau^2},$$

$$\mu p^2 = \left(\mu - \frac{\pi^2}{4\tau^2}\right) r^2 + \frac{\pi^2 c^2}{4\tau^2},$$

which is the differential equation to the curve.

Euler; *Mechan.* Tom. II. p. 208.

(8) An infinite number of similar curves originate at a given point; to determine the corresponding synchronous curve, or the curve which shall cut them in such a manner that a particle acted on by gravity may describe the intercepted arcs in equal times.

Let O (fig. 153) be the given point, and CPD the synchronous curve intercepting the arc OP of the curve OPQ, which is one of the similar curves. Let Ox, a vertical line, be taken as the axis of x, and Oy, at right angles to it, as the axis of y. Let $OM = x$, $PM = y$, $OP = s$. Then, if k denote the time down OP, which by hypothesis is constant for every point P in the curve CPD, we have

$$k = \int_0^s \frac{ds}{(2gx)^{\frac{1}{2}}} = \int_0^x \frac{(1+p^2)^{\frac{1}{2}}}{(2gx)^{\frac{1}{2}}}\, dx \ldots\ldots\ldots\ldots (1),$$

where p is equal to $\dfrac{dy}{dx}$.

Now, by the nature of similar curves, the equation to the curve OPQ is of the form

$$F\left(\frac{x}{a}, \frac{y}{a}\right) = 0, \quad \text{or} \quad y = af\left(\frac{x}{a}\right) \ldots\ldots\ldots\ldots (2),$$

where F, f, denote certain functions of the quantities to which they are prefixed, a being the value of the general parameter of the class of similar curves for the individual curve OPQ. Hence, assuming

$$x = a\tau, \text{ and therefore } y = af(\tau), \text{ by (2)}, \ldots\ldots\ldots\ldots \text{(3)},$$

we have from (1),

$$k = a^{\frac{1}{2}} \int_0^\tau \frac{(1 + T^2)^{\frac{1}{2}}}{(2g\tau)^{\frac{1}{2}}} \, d\tau = a^{\frac{1}{2}} \phi(\tau) \ldots\ldots\ldots\ldots\ldots \text{(4)},$$

where $T = \dfrac{dy}{dx} = \dfrac{d}{d\tau} f(\tau)$, and $\phi(\tau)$ is some function of τ. Hence, from (3) and (4), there is

$$x = \frac{k^2 \tau}{\{\phi(\tau)\}^2}, \quad y = \frac{k^2 f(\tau)}{\{\phi(\tau)\}^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}.$$

Eliminating τ between these two last equations, we shall obtain an equation in x, y, the required equation to the synchronous curve.

If the integration indicated in the equation (1) can be effected, then it is needless to have recourse to the subsidiary symbol τ. We have merely in this case to eliminate, after the performance of the integration, the parameter a, by the aid of the equation (2). It rarely happens, however, that we can execute the operation of integration, and under these circumstances the equations (5) will enable us to construct the synchronous curve by the method of quadratures; a pair of values of x, y, and therefore a point in the synchronous curve, being ascertained approximately for every numerical value which we may assign to τ.

The problem of Synchronous Curves was first discussed by John Bernoulli, in the *Act. Erudit. Lips.* 1697, Mai. p. 206. The subject was afterwards investigated by Saurin, and by Euler[1].

(9) An assemblage of circles in the plane xOy, (fig. 153), all touch Ox in the point O; to determine the synchronous curve, Ox being vertical, and gravity the accelerating force; the descent being supposed to commence from O.

[1] *Mechan.* Tom. II. p. 47; *Mém. de l'Acad. de St. Pétersb.* 1819—1820, pp. 20, 35.

The equation to any one of the circles, its radius being a, will be

$$x^2 = 2ay - y^2,$$

or

$$y = a\left\{1 - \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}\right\}.$$

Adopting the notation of the preceding general problem, we have

$$f(\tau) = 1 - (1 - \tau^2)^{\frac{1}{2}},$$

$$T = \frac{df(\tau)}{d\tau} = \frac{\tau}{(1 - \tau^2)^{\frac{1}{2}}},$$

$$\phi(\tau) = \int_0^\tau \frac{(1 + T^2)^{\frac{1}{2}}}{(2g\tau)^{\frac{1}{2}}} d\tau = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}.$$

Hence $$x = \frac{2gk^2\tau}{\left\{\int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}\right\}^2}, \quad y = \frac{2gk^2\{1 - (1 - \tau^2)^{\frac{1}{2}}\}}{\left\{\int_0^\tau \frac{d\tau}{(\tau - \tau^3)^{\frac{1}{2}}}\right\}^2},$$

whence the required curve may be constructed by the method of quadratures. Euler; *Mechan.* Tom. II. p. 52.

(10) A particle acted on by any assigned accelerating forces in one plane moves along a curve from one given point to another; to determine the form of the curve that the whole time of the motion between the two points may be the least possible.

Let P (fig. 154) be any point in the required curve; let $OM = x$, $PM = y$; let A and B be the two given points; let $AP = s$; also let α, β, be the values of x at the points A, B. Then, v being the velocity of the particle at P,

$$dt = \frac{ds}{v} = \frac{(1 + p^2)^{\frac{1}{2}}}{v} dx, \quad \text{where } p = \frac{dy}{dx},$$

and the whole time from A to B will be equal to

$$\int_\alpha^\beta \frac{(1 + p^2)^{\frac{1}{2}}}{v} dx \text{.........................} (1).$$

Assume $\frac{(1 + p^2)^{\frac{1}{2}}}{v} = V$; then, that the expression (1) may be a minimum, we have, by the Calculus of Variations, since V in-

volves only p and v, of which the latter is a function of only x and y,

$$N - \frac{dP}{dx} = 0 \dots\dots\dots\dots\dots (2),$$

where N, P, denote respectively the partial differential coefficients of V with regard to y, p; $\frac{dP}{dx}$ representing the total differential coefficient of P with respect to x.

Now $$N = -\frac{1}{v^2}(1+p^2)^{\frac{1}{2}}\frac{dv}{dy} \dots\dots\dots\dots\dots (3),$$

where $\frac{dv}{dy}$ signifies the partial differential coefficient of v with regard to y; but

$$vdv = Xdx + Ydy \dots\dots\dots\dots\dots (4),$$

where X, Y, represent the resolved parts of the whole accelerating force on the particle parallel to Ox, Oy; and therefore, in (3), $\frac{dv}{dy} = \frac{Y}{v}$. Hence

$$N = -\frac{Y}{v^3}(1+p^2)^{\frac{1}{2}} = -\frac{Y}{v^3}\frac{ds}{dx}.$$

Again, $$P = \frac{p}{v(1+p^2)^{\frac{1}{2}}} = \frac{1}{v}\frac{dy}{ds}.$$

Hence, substituting for N and P in (2), we have

$$\frac{Y}{v^3}\frac{ds}{dx} + \frac{d}{dx}\left(\frac{1}{v}\frac{dy}{ds}\right) = 0,$$

$$\frac{Y}{v^3}\frac{ds}{dx} - \frac{1}{v^2}\frac{dv}{dx}\frac{dy}{ds} + \frac{1}{v}\frac{d}{dx}\frac{dy}{ds} = 0.$$

But, from (4), $$\frac{dv}{dx} = \frac{1}{v}\left(X + Y\frac{dy}{dx}\right).$$

Hence $$\frac{Y}{v^3}\frac{ds}{dx} - \frac{1}{v^3}\left(X + Y\frac{dy}{dx}\right)\frac{dy}{ds} + \frac{1}{v}\frac{d}{dx}\frac{dy}{ds} = 0;$$

and therefore, after a few obvious simplifications,

$$v^2\frac{d}{dx}\frac{dy}{ds} = X\frac{dy}{ds} - Y\frac{dx}{ds} \dots\dots\dots\dots\dots (5).$$

If from this equation we eliminate v by the aid of (3), we shall obtain a differential equation of the second order, which is the equation to the required curve. The two arbitrary constants introduced by the integration are to be determined from the conditions that the curve shall pass through the two given points A and B.

The equation (5) is equivalent to

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} - X\frac{dy}{ds},$$

where ρ denotes the radius of curvature at the point P; a result which shews that the pressure on the curve due to the centrifugal force is equal to that which arises from the accelerating forces which act upon the particle.

The curve in question belongs to a class of mechanical curves called Brachystochrones, which are characterized by the general property that a particle under the action of assigned accelerating forces, shall move along them between given limits in the least time possible.

The problem of the Brachystochrone between two given points, when gravity is the accelerating force, was proposed by John Bernoulli[1], as a challenge to the mathematicians of the day. Six months was the time allotted for its solution. Leibnitz[2] was immediately successful, and communicated his good fortune by letter to Bernoulli. No other solution however having made its appearance within the prescribed time, Bernoulli, in conformity with the desire of Leibnitz, consented to prorogue the term of the challenge to the following Easter, the results obtained by himself and Leibnitz being suppressed for that interval. A programme was accordingly published at Groningen, in January 1697, again announcing the problem and repeating the challenge. In consequence of this delay solutions were obtained by three other mathematicians: by Newton[3], anonymously; by James Bernoulli[4]; and by L'Hôpital[5]. The solution of Leibnitz was

[1] *Act. Erudit. Lips.* 1696, Jun. p. 269.
[2] *Commerc Epistol.* Leibnitii et Bernoullii, Epist. 28.
[3] *Phil. Trans.* 1697, Num. 224, p. 389.
[4] *Act Erudit. Lips.* Mai. 169[illegible], p. 212.
[5] *Act Erudit. Lips.* ib. 217.

announced in the *Acta Erudit. Lips.* Mai. 1697, p. 203. The conclusions of Newton, Leibnitz, and L'Hôpital were given without the analysis. John Bernoulli gave two different solutions, one direct and the other indirect. The latter was published in the *Acta Erudit. Lips.* Mai. 1697, p. 207; the former was not made public till the year 1718, in a Memoir on Isoperimetrical problems, in the *Mémoires de l'Académie des Sciences de Paris*, p. 136; see also his works, Tom. II. p. 266. A solution of the problem was afterwards given by Craig[1], who had merely seen Newton's result without consulting the analysis which had been given by John and by James Bernoulli.

(11) To find the brachystochrone when a particle, acted on by a central force attracting with an intensity which varies inversely as the square of the distance, moves along a curve from one given point to another.

Let A (fig. 155) be the point where the motion commences, and B the point at which the particle is to arrive in the shortest time possible. Let P be any point in the brachystochrone, S the centre of force; let $SP=r$, $p=$ the perpendicular from S upon the tangent at P, $SA=a$, $\phi=$ the angle between SP and the tangent at P, $\mu=$ the absolute force of attraction, $v=$ the velocity, and $\rho=$ the radius of curvature at P. Then, since the pressure on the curve due to the centrifugal force must be equal to that due to the attraction, we have

$$\frac{v^2}{\rho}=\frac{\mu}{r^2}\sin\phi \quad\text{.........................}\quad (1).$$

But
$$v^2=C-2\int\frac{\mu}{r^2}\,dr=C+\frac{2\mu}{r},$$

or, since $v=0$ when $r=a$,

$$v^2=2\mu\left(\frac{1}{r}-\frac{1}{a}\right) \quad\text{......................}\quad (2).$$

Also, the curve being convex towards S,

$$\rho=-r\frac{dr}{dp} \quad\text{..............................}\quad (3).$$

[1] *Phil. Trans.* 1701, Vol. XXII. p. 746.

From (1), (2), (3), we have

$$\frac{2\mu\left(\frac{1}{a}-\frac{1}{r}\right)}{r\frac{dr}{dp}}=\frac{\mu}{r^2}\frac{p}{r},$$

$$2\left(\frac{1}{a}-\frac{1}{r}\right)=\frac{p}{r^2}\frac{dr}{dp},\quad 2\frac{dp}{p}=\frac{adr}{r(r-a)};$$

integrating, we get

$$2\log p=\log C+\log\frac{r-a}{r},$$

$$\log p^2=\log\left(C\frac{r-a}{r}\right);$$

but C must be a negative quantity, because, as will appear from the equation (2), a is greater than r; hence, putting $-A$ for C, we have, for the differential equation to the brachystochrone,

$$p^2=A\frac{a-r}{r}.$$

If from this equation we were to obtain, by integration, a relation between r and an angular co-ordinate θ, we should introduce another constant into the equation in addition to A. Both these constants would have to be determined by the conditions that the curve must pass through both A and B.

Euler; *Mechan.* Tom. II. p. 191.

(12) To find the inclination of a thin tube to the horizon, so that a descending particle may describe the greatest horizontal space in a given time.

The required angle of inclination $=45°$.

(13) A particle having been placed at the point A, (fig. 156), moves along a thin tube APS towards a centre of attractive force in S which varies as any function of the distance; to find the nature of the curve of the tube that the time through any arc AP may be n times as great as through a portion Ap of the prime radius vector SA, Sp being equal to SP.

Let $SP=r$, $SA=a$, $\angle ASP=\theta$; then the equation to the curve will be

$$r=a\,.\,\epsilon^{-\frac{\theta}{(n^2-1)^{\frac{1}{2}}}}.$$

(14) A particle is projected with a given velocity from a point A (fig. 157) along a curve APO in which it is constrained to move, and is acted upon by a force always tending to O, and varying directly as the distance; to find the nature of this curve in order that the angular velocity of the radius vector OP may be invariable.

Let $AO=a$, $OP=r$, $\angle AOP=\theta$, $\mu^2=$ the absolute force of attraction, $\omega=$ the angular velocity of OP, $\beta=$ the initial velocity of the particle; then the equation to the curve will be

$$\left(\frac{\mu^2+\omega^2}{\omega^2}\right)^{\frac{1}{2}}\theta=\cos^{-1}\left\{\left(\frac{\mu^2+\omega^2}{\mu^2a^2+\beta^2}\right)^{\frac{1}{2}}r\right\}-\cos^{-1}\left\{\left(\frac{\mu^2+\omega^2}{\mu^2a^2+\beta^2}\right)^{\frac{1}{2}}a\right\}.$$

Euler; *Mechan.* Tom. II. p. 138.

(15) A particle is acted on by an attractive force tending to a centre, and varying inversely as the square of the distance; to find the tautochrone.

If τ denote the time of the motion, and the notation remain the same as in problem (7), the differential equation to the tautochrone will be

$$p^2=r^2-\frac{\pi^2\tau^2}{2\mu c}(r-c)r^5.$$

Euler; *Mechan.* Tom. II. p. 209.

(16) To find the tautochrone when the central attractive force is constant.

If f denote the constant central force, the equation to the tautochrone will be

$$p^2=\left(1+\frac{\pi^2\tau^2c}{2f}\right)r^2-\frac{\pi^2\tau^2}{2f}r^3.$$

Euler; *Mechan.* Tom. II. p. 210.

(17) An infinite number of straight lines originate at a single point and lie in one plane; to determine the synchronous curve, gravity being the accelerating force.

The given point being taken as the origin of co-ordinates, the axis of x extending vertically downwards, and that of y being horizontal; the synchronous curve will be a circle of which the equation is

$$x^2 + y^2 = \tfrac{1}{2} g k^2 x,$$

where k denotes the common time of descent.

Euler; *Mém. de l'Acad. de St. Pétersb.* 1819, 1820, p. 22.

(18) There is an infinite number of cycloids, of which the bases all commence at the origin of co-ordinates, and coincide with the axis of y, which is horizontal; to find the synchronous curve, gravity being the accelerating force, and the motion commencing from the origin.

Let k denote the constant time of descent; then, the axis of x being vertical, the equation to the required curve depends upon the elimination of a between the two equations

$$x = a \operatorname{vers} \left\{ k \left(\frac{g}{a} \right)^{\frac{1}{2}} \right\}, \qquad y = a \operatorname{vers}^{-1} \frac{x}{a} - (2ax - x^2)^{\frac{1}{2}},$$

and will cut all the cycloids at right angles.

John Bernoulli; *Act. Erudit. Lips.* 1697, Mai. p. 206.

(19) A particle, acted on by a central force attracting directly as the distance, moves along a curve from one given point to another; to find the nature of the curve when it is brachystochronous.

Let A (fig. 155) be the point where the motion commences, and B the point where the particle is to arrive in the shortest time possible. Let P be any point in the brachystochrone; let $SP = r$, $p =$ the perpendicular from S, the centre of force, upon the tangent at P, $a = SA$. Then the equation to the curve between p and r will be

$$p^2 = A (r^2 - a^2),$$

where A is a constant quantity, which is the equation to the hypocycloid.

If from this equation we were to obtain by integration a relation between r and an angular co-ordinate θ, we should have

another constant in the equation in addition to A. Both these constants would have to be determined by the conditions that the curve must pass through both A and B.

Euler; *Mechan.* Tom. II. p. 191.

SECT. 4. *Inverse Problems on the Pressure of a Particle on Smooth Fixed Curves.*

(1) A particle descends down a curve line in a vertical plane by the action of gravity; to find the nature of the curve that the pressure may be invariable.

Let OA (fig. 158) be the required curve; Ox, vertical, the axis of x, Oy, horizontal, the axis of y; P any point in the curve; let $OM=x$, $PM=y$, $OP=s$; let k be the constant pressure; β the initial velocity of the particle, O being its initial position. Then, by formula (A) of Sect. (II.), we have

$$k = g\frac{dy}{ds} + \frac{1}{\rho}\frac{ds^2}{dt^2} \text{ ....(1)},$$

where ρ denotes the magnitude of the radius of curvature at P.

But $$\frac{ds^2}{dt^2} = \beta^2 + 2gx;$$

also, s being taken as the independent variable,

$$\frac{1}{\rho} = \frac{\frac{d^2y}{ds^2}}{\frac{dx}{ds}}.$$

Hence, from (1), we have

$$k = g\frac{dy}{ds} + (2gx+\beta^2)\frac{\frac{d^2y}{ds^2}}{\frac{dx}{ds}},$$

$$\frac{k}{(2gx+\beta^2)^{\frac{1}{2}}}\frac{dx}{ds} = (2gx+\beta^2)^{\frac{1}{2}}\frac{d^2y}{ds^2} + \frac{g}{(2gx+\beta^2)^{\frac{1}{2}}}\frac{dy}{ds}\frac{dx}{ds}.$$

Integrating, we have

$$\frac{k}{g}(2gx+\beta^2)^{\frac{1}{2}} = (2gx+\beta^2)^{\frac{1}{2}}\frac{dy}{ds} + C,$$

$$\frac{dy}{ds} = \frac{k}{g} - \frac{C}{(2gx+\beta^2)^{\frac{1}{2}}},$$

where C is an arbitrary constant. Assume α to be the inclination of the curve to the vertical at the origin; then

$$\sin\alpha = \frac{k}{g} - \frac{C}{\beta};$$

and therefore,

$$\frac{dy}{ds} = \frac{k}{g} - \frac{\beta}{g}\frac{k-g\sin\alpha}{(2gx+\beta^2)^{\frac{1}{2}}} \qquad (2).$$

The relation between x and y may be obtained by a second integration, but the result is of little value in consequence of its complexity. For the investigation of the form of the curve which corresponds to the differential equation (2), the reader is referred to Whewell's *Dynamics*, part II. p. 95; or, Earnshaw's *Dynamics*, p. 129.

The problem of the Curve of Equal Pressure, in the case of gravity, was first proposed by John Bernoulli[1], and solved by L'Hôpital[2]. Various problems of a similar character were afterwards discussed by Varignon[3].

Commerc. Epistolic. Leibnitii et Bernoullii, Epist. VII.

(2) A particle, acted on by gravity, descends from a point O (fig. 158) down a curve OA, which it presses at each point of its descent with a force varying as the square of its distance below the horizontal line through O; to find the nature of the curve OA, the initial velocity of the particle being zero.

Let the axes Ox, Oy, be taken vertical and horizontal; let k be the pressure on the curve when x is equal to unity. Then, by the formula (A) of Sect. (II.),

[1] *Act. Erudit. Suppl.* Tom. II. sect. vi. p. 291.
[2] *Mém. de l'Acad. des Sciences de Paris*, 1700, p. 9.
[3] *Mém. de l'Acad. des Sciences de Paris*, 1710, p. 196.

$$kx^2 = g\frac{dy}{ds} + \frac{2gx}{\rho};$$

but
$$\rho = -\frac{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}};$$

hence
$$kx^2 = \frac{g}{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{1}{2}}} - \frac{2gx\dfrac{d^2x}{dy^2}}{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}},$$

$$kx^{\frac{3}{2}}\frac{dx}{dy} = \frac{gx^{-\frac{1}{2}}\dfrac{dx}{dy}}{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{1}{2}}} - \frac{2gx^{\frac{1}{2}}\dfrac{dx}{dy}\dfrac{d^2x}{dy^2}}{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}};$$

integrating, we have

$$\tfrac{2}{5}kx^{\frac{5}{2}} = \frac{2gx^{\frac{1}{2}}}{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{1}{2}}},$$

no constant being added because the curve passes through the origin. Putting $\frac{5g}{k} = a^2$, we get

$$x^2\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}} = a^2,$$

$$(a^4 - x^4)^{\frac{1}{2}}\,dy = x^2 dx,$$

which is the equation to the Elastic Curve of James Bernoulli[1].

Varignon; *Mémoires de l'Académie des Sciences de Paris*, 1710, p. 151.

(3) A particle, acted upon by a force parallel to the axis of x, is constrained to move along a given curve OPA (fig. 158); to find the law of the force that the curve may experience an invariable pressure.

Let k denote the constant pressure, β the velocity of the particle at O, which we will take as the origin of co-ordinates,

[1] *Act. Erudit. Lips.* 1694, p. 272; 1695, p. 538.

and X the force, at any point P of the curve, parallel to Ox. Then, by formula (A) of Section (II.) and formula (D) of Section (I.), we have

$$k = X\frac{dy}{ds} + \frac{1}{\rho}\{\beta^2 + 2\int_0^x X dx\}.$$

Taking s as the independent variable, we have

$$\frac{1}{\rho} = \frac{\frac{d^2y}{ds^2}}{\frac{dx}{ds}};$$

and the equation becomes

$$k\frac{dx}{ds} = \beta^2\frac{d^2y}{ds^2} + X\frac{dx}{ds}\frac{dy}{ds} + 2\frac{d^2y}{ds^2}\int_0^x X dx.$$

Multiplying by $\frac{dy}{ds}ds$, and integrating

$$k\int\frac{dx}{ds}\frac{dy}{ds}ds = \tfrac{1}{2}\beta^2\frac{dy^2}{ds^2} + \frac{dy^2}{ds^2}\int_0^x X dx,$$

$$\int_0^x X dx = -\tfrac{1}{2}\beta^2 + \frac{k}{\frac{dy^2}{ds^2}}\int\frac{dx}{ds}\frac{dy}{ds}ds;$$

and therefore, putting $\frac{dy}{dx} = p$,

$$\int_0^x X dx = -\tfrac{1}{2}\beta^2 + k\left(\frac{1}{p^2}+1\right)\left\{\int\frac{p\,dx}{(1+p^2)^{\frac{1}{2}}} + C\right\}\dots\dots\dots\dots(1).$$

Differentiating with respect to x, we obtain the required expression for the force

$$X = \frac{k}{p}(1+p^2)^{\frac{1}{2}} - \frac{2k}{p^3}\frac{dp}{dx}\left\{\int\frac{p\,dx}{(1+p^2)^{\frac{1}{2}}} + C\right\}\dots\dots\dots\dots(2).$$

If we put $x = 0$, we have, from (1),

$$2k\left(\frac{1}{p^2}+1\right)\left\{\int\frac{p\,dx}{(1+p^2)^{\frac{1}{2}}} + C\right\} = \beta^2,$$

a condition which will determine the value of the arbitrary constant C.

Euler; *Mechan.* Tom. II. p. 101.

(4) A particle moves along a parabola OA, of which the axis is Oy, under the action of a force always parallel to Ox, which is at right angles to Oy; to determine the law of the force that the particle may exert the same pressure on the curve during the whole of its motion.

Let Ox, Oy, be the co-ordinate axes, k the constant pressure, and $x^2 = ay$ the equation to the parabola. Then, by the formula for X given in the preceding problem, since $p = \dfrac{2x}{a}$, we have

$$X = \frac{ka}{2x}\left(1 + \frac{4x^2}{a^2}\right)^{\frac{1}{2}} - \frac{ka^2}{2x^3}\left\{\int \frac{2x\,dx}{(a^2+4x^2)^{\frac{1}{2}}} + C\right\}$$

$$= \frac{k}{2x}(a^2+4x^2)^{\frac{1}{2}} - \frac{ka^2}{2x^3}\{\tfrac{1}{2}(a^2+4x^2)^{\frac{1}{2}} + C\},$$

where C is a constant quantity,

$$= -\frac{ka^2}{2x^3}C - \frac{k}{4x^3}(a^2-2x^2)(a^2+4x^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots.(1).$$

Again, by the formula (1) in the preceding problem,

$$\int_0^x X\,dx = -\tfrac{1}{2}\beta^2 + k\left(\frac{a^2}{4x^2}+1\right)\{\tfrac{1}{2}(a^2+4x^2)^{\frac{1}{2}} + C\},$$

$$4x^2\int_0^x X\,dx = -2\beta^2x^2 + k(a^2+4x^2)\{\tfrac{1}{2}(a^2+4x^2)^{\frac{1}{2}} + C\};$$

hence, putting $x = 0$, we get

$$0 = \tfrac{1}{2}a + C,$$

and therefore, by (1),

$$X = \frac{ka^3}{4x^3} - \frac{k}{4x^3}(a^2-2x^2)(a^2+4x^2)^{\frac{1}{2}}.$$

Euler; *Mechan.* Tom. II. p. 103.

(5) A particle, under the action of gravity, descends from a point O down a curve OA, (fig. 159), which it presses, at each point of its descent, with a force varying as its perpendicular distance from the horizontal line through O; to find the nature of the curve OA, the initial velocity of the particle being zero.

Take Ox, Oy, the axes of co-ordinates, vertical and horizontal; let k be the pressure on the curve when x is equal to unity; then, putting $a = \frac{g}{k}$, the equation to the curve will be

$$x^2 = 6ay - y^2,$$

the equation to a circle of which OE the diameter is equal to $6a$.

Varignon; *Mém. de l'Acad. des Sciences de Paris*, 1710, p. 151.

(6) To find the curve when the pressure varies as the square root of the distance.

The equation to the curve is

$$y = 2a \operatorname{vers}^{-1} \frac{x}{2a} - (4ax - x^2)^{\frac{1}{2}},$$

which belongs to a cycloid OBA, (fig. 160), the radius of the generating circle being $2a$.

Varignon; *Ib.* p. 152.

(7) A particle, acted on by gravity, descends from rest down a curve; to find the nature of the curve that the pressure at any point due to the centrifugal force may vary as any power of the distance of the particle below the horizontal line passing through its initial position.

Let k denote the pressure due to centrifugal force when x is equal to unity, the axis of x being vertical, as in fig. 158; then the differential equation to the curve will be, putting $\frac{g}{k} = a$,

$$(4n^2a^2 - x^{2n})^{\frac{1}{2}}\, dy = x^n\, dx.$$

Varignon; *Ib.* p. 156.

(8) To find the curve when the part of the pressure which is due to gravity varies as the n^{th} power of the depth of the descent.

The notation being the same as in the preceding problem, except that k denotes the pressure due to gravity alone when $x = 1$, the differential equation to the curve will be

$$(a^2 - x^{2n})^{\frac{1}{2}}\, dy = x^n dx.$$

Varignon; *Ib.* p. 160.

(9) A particle descends from rest by the action of gravity down a curve line; to determine the nature of the curve when the part of the pressure due to centrifugal force bears a constant ratio to that due to gravity.

Let Oy (fig. 158) be horizontal and Ox vertical; then, if $\frac{m}{n}$ denote the constant ratio, the differential equation to the curve will be

$$(a^{\frac{m}{n}} - x^{\frac{m}{n}})^{\frac{1}{2}}\, dy = x^{\frac{m}{2n}}\, dx,$$

where a is a constant quantity. If $m = n$, the curve will be an inverted cycloid with its base horizontal.

Varignon; *Ib.* p. 161.

(10) A particle, acted on by a force parallel to Ox, (fig. 160) moves from rest along the arc OB of a cycloid; to determine this force that the curve may always experience the same pressure.

If k denote the constant pressure, a the radius of the generating circle, X the required force, and $Om = x$; then

$$X = \frac{k}{3}\left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

Euler; *Mechan.* Tom. II. p. 104.

(11) A particle is projected along a smooth groove from a point which is half way between two centres of force of equal intensity, each varying inversely as the distance: to find what the form of the groove must be in order that the particle may move uniformly.

If a denote the initial distance of the particle from each centre, and r, r', the distances of any point in the curve from the two centres,

$$r \,.\, r' = a^2.$$

SECT. 5. *Motion of Particles acted on by smooth constraining lines moveable according to assigned geometrical conditions.*

Let Ox (fig. 161) be the axis of x, and Oy, at right angles to it, that of y. Let P be the position of the particle in the plane xOy

at any time t from the commencement of the motion; let $OM=x$, $PM=y$. Let X, Y, denote the resolved parts of the accelerating forces on the particle parallel to the axes of co-ordinates, and X', Y', the resolved parts of the force of constraint at any point of its path. Then the circumstances of its motion will depend upon the differential equations

$$\frac{d^2x}{dt^2}=X+X', \qquad \frac{d^2y}{dt^2}=Y+Y' \ldots\ldots\ldots\ldots(\text{A}).$$

The complete solution of the general problem of the motion of the particle consists in the determination of x and y in terms of t. But the equations (A) involve, in addition to x and y, the two unknown quantities X' and Y'. Hence it appears that the general consideration of the motion affords us only two equations involving four unknown quantities. From this it is clear that the analytical expression of the conditions to which the motion of the constraining line in any particular problem is subject, must be virtually equivalent to two more equations involving only x, y, X', Y'.

Let r be the distance PO of the particle at the time t from the origin of co-ordinates, and let $\angle POx=\theta$; then, in case the action of the constraining line always takes place in a direction at right angles to OP, and F denote the sum of the resolved parts of the accelerating forces on the particle along OP, we may obtain from the equations (A) the formula

$$\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2}=F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B}).$$

The formula (B) was given by Ampère, *Annales de Gergonne*, Tom XX. p. 37 et sq.

(1) To find the path of a particle upon a smooth horizontal plane, fastened by a thread to a point of which the motion is uniform and rectilinear in that plane.

Let Q (fig. 162) be constrained to move uniformly along the line Ox, and let P be the position of the particle in the plane xOy at any time t; PQ being the thread by which P is attached to Q. Let $OM=x$, $PM=y$, $OQ=x'$, $PQ=h$, $\angle PQO=\theta$. Then, R denoting the tension of the thread, we have, by the equations

(A), since $X' = \frac{R}{m'}\cos\theta$, $Y' = -\frac{R}{m'}\sin\theta$, where m' denotes the mass of the particle,

$$\frac{d^2x}{dt^2} = \frac{R}{m'}\cos\theta, \qquad \frac{d^2y}{dt^2} = -\frac{R}{m'}\sin\theta.$$

Hence, eliminating R,

$$\sin\theta\frac{d^2x}{dt^2} + \cos\theta\frac{d^2y}{dt^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

again, it is clear that

$$x' = x + h\cos\theta = mt + n \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where m denotes the uniform velocity of Q, and n its initial distance from O; and therefore

$$\frac{d^2x}{dt^2} + h\frac{d^2}{dt^2}\cos\theta = 0;$$

also, by the geometry,

$$y = h\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and therefore
$$\frac{d^2y}{dt^2} = h\frac{d^2}{dt^2}\sin\theta.$$

Hence, from (1),

$$\cos\theta\frac{d^2}{dt^2}\sin\theta - \sin\theta\frac{d^2}{dt^2}\cos\theta = 0;$$

integrating, we obtain

$$\cos\theta\frac{d}{dt}\sin\theta - \sin\theta\frac{d}{dt}\cos\theta = \omega, \quad \text{or } \frac{d\theta}{dt} = \omega,$$

where ω is a constant quantity, which shews that the angular velocity of the thread PQ about Q is invariable.

Integrating again, $$\theta = \alpha + \omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$
α being the initial value of θ.

Hence we have, from (2) and (3),

$$x = mt + n - h\cos(\alpha + \omega t),$$
$$y = h\sin(\alpha + \omega t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

which give the values of x and y at any time during the motion.

Eliminating t, we get, as the equation to the path of the particle in rectangular co-ordinates,

$$x = \frac{m}{\omega} \sin^{-1} \frac{y}{h} - (h^2 - y^2)^{\frac{1}{2}} + n - \frac{m\alpha}{\omega}.$$

The equations (2) and (4) however furnish us with the most convenient conception of the motion of the particle. In fact they shew that P's motion may be perfectly represented by supposing it to move with a uniform velocity ωh in the circumference of a circle of which the radius is h, and of which the centre moves along Ox with a uniform velocity m. The path of P is therefore a trochoid.

From (5), we have, by differentiation,

$$\frac{d^2y}{dt^2} = - h\omega^2 \sin (\alpha - \omega t).$$

Hence, from (4), and the original equations of motion,

$$- h\omega^2 \sin (\alpha + \omega t) = - \frac{R}{m'} \sin (\alpha + \omega t),$$

and therefore $$\frac{R}{m'} = h\omega^2, \quad R = m'h\omega^2,$$

which shews that the tension of the string is invariable.

This is an example of a class of curves called Tractories, which are traced by a material particle attached to one extremity of a string while the other is constrained to move along some assigned curve with a given velocity. The curve in which the end of the string is constrained to move is called the Directrix.

This problem formed the subject of a controversy between Fontaine and Clairaut; the solution given by Fontaine depended upon the assumption that the string would be always a tangent to the path of the particle, an hypothesis which Clairaut declared to be erroneous, and which, in fact, virtually involves a neglect of inertia. Fontaine's assumption would be admissible for the motion of a particle on a perfectly rough plane, where its motion would be destroyed the moment it was generated.

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1736, p. 4. Euler; *Nova Acta Acad. Petrop.* 1784.

(2) A thin rectilinear tube is constrained to move in a horizontal plane round a vertical axis passing through one extremity with a uniform angular velocity: to find the motion of a particle sliding freely within the tube.

By the formula (B), since no accelerating forces act on the particle,

$$\frac{d^2r}{dt^2} = r\frac{d\theta^2}{dt^2} = \omega^2 r,$$

where ω denotes the invariable angular velocity.

The integral of this equation is

$$r = A\epsilon^{\omega t} + B\epsilon^{-\omega t} \dots\dots\dots\dots\dots\dots\dots(1),$$

where A and B are constants.

Let a, β, be the initial values of r, $\frac{dr}{dt}$; then, from (1),

$$a = A + B,$$

and

$$\beta = A\omega - B\omega;$$

and therefore

$$A = \frac{1}{2\omega}(\omega a + \beta),$$

$$B = \frac{1}{2\omega}(\omega a - \beta).$$

Hence, putting the values of A and B in (1), we see that

$$2\omega r = (\omega a + \beta)\,\epsilon^{\omega t} + (\omega a - \beta)\,\epsilon^{-\omega t},$$

which gives the value of r at any time during the motion.

The equation to the path of the particle is, putting $\theta = \omega t$,

$$2\omega r = (\omega a + \beta)\,\epsilon^{\theta} + (\omega a - \beta)\,\epsilon^{-\theta}.$$

John Bernoulli; *Opera*, Tom. IV. p. 248. Clairaut; *Mém. Acad. Paris*, 1742, p. 10.

(3) A material particle is placed within a thin circular tube, which is constrained to revolve with a uniform angular velocity in a horizontal plane about a point in its circumference; to investigate the motion of the particle.

Let O (fig. 163) be the point about which the circle APO is constrained to revolve; C its centre at any time t, and P the position of the particle; R the action of the circle on the particle, which will take place in the direction PC. Let Ox, Oy, be the axes of co-ordinates, x, y, being the co-ordinates of P.

Let $\angle POx = \theta$, $\angle OPC = \angle COP = \phi$, $OP = r$, $OC = a$.

Then, since no accelerating forces act on the particle, we have, by the formulæ (A),

$$\frac{d^2x}{dt^2} = -R\cos(\theta - \phi),$$

$$\frac{d^2y}{dt^2} = -R\sin(\theta - \phi):$$

multiplying these equations by $\sin(\theta - \phi)$, $\cos(\theta - \phi)$, and subtracting, we have

$$\sin(\theta - \phi)\frac{d^2x}{dt^2} - \cos(\theta - \phi)\frac{d^2y}{dt^2} = 0,$$

or, since $x = r\cos\theta$, $y = r\sin\theta$,

$$\sin(\theta - \phi)\frac{d^2}{dt^2}(r\cos\theta) - \cos(\theta - \phi)\frac{d^2}{dt^2}(r\sin\theta) = 0.$$

But, from the geometry, it is evident that

$$r = 2a\cos\phi:$$

hence

$$\sin(\theta - \phi)\frac{d^2}{dt^2}(\cos\theta\cos\phi) - \cos(\theta - \phi)\frac{d^2}{dt^2}(\sin\theta\cos\phi) = 0,$$

$$\sin(\theta - \phi)\frac{d^2}{dt^2}\{\cos(\theta + \phi) + \cos(\theta - \phi)\}$$

$$-\cos(\theta - \phi)\frac{d^2}{dt^2}\{\sin(\theta + \phi) + \sin(\theta - \phi)\} = 0.$$

But, supposing ω to be the angular velocity of the diameter OCA of the circle about O, and $\angle AOx$ to be initially zero, it is clear that

$$\angle AOx \text{ or } \theta + \phi = \omega t \text{.....................} (1).$$

Hence, putting ωt for $\theta+\phi$,

$$\sin(\theta-\phi)\frac{d^2}{dt^2}\cos(\theta-\phi)-\cos(\theta-\phi)\frac{d^2}{dt^2}\sin(\theta-\phi)$$

$$+\sin(\theta-\phi)\frac{d^2}{dt^2}\cos\omega t-\cos(\theta-\phi)\frac{d^2}{dt^2}\sin\omega t=0,$$

$$\frac{d}{dt}\left\{\sin(\theta-\phi)\frac{d}{dt}\cos(\theta-\phi)-\cos(\theta-\phi)\frac{d}{dt}\sin(\theta-\phi)\right\}$$

$$-\omega^2\sin(\theta-\phi)\cos\omega t+\omega^2\cos(\theta-\phi)\sin\omega t=0,$$

$$-\frac{d^2}{dt^2}(\theta-\phi)-\omega^2\sin(\theta-\phi-\omega t)=0.$$

But, by (1), we have $\theta=\omega t-\phi$; hence

$$2\frac{d^2\phi}{dt^2}+\omega^2\sin 2\phi=0.$$

Multiplying by $2\dfrac{d\phi}{dt}$, and integrating,

$$2\frac{d\phi^2}{dt^2}-\omega^2\cos 2\phi=C\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

For the sake of simplicity we will suppose that, initially, P coincides with A, and that its velocity is zero; hence, when $t=0$, we have $\phi=0$, and since, from (1),

$$\frac{d\theta}{dt}+\frac{d\phi}{dt}=\omega,$$

we have also initially $\dfrac{d\phi}{dt}=\omega$. Hence, from (2),

$$\omega^2=C,$$

and therefore
$$2\frac{d\phi^2}{dt^2}-\omega^2\cos 2\phi=\omega^2,$$

$$2\frac{d\phi^2}{dt^2}=\omega^2(1+\cos 2\phi)=2\omega^2\cos^2\phi,$$

$$\frac{d\phi}{dt}=\omega\cos\phi,$$

$$\omega dt=\frac{d\phi}{\cos\phi}=\frac{\cos\phi d\phi}{1-\sin^2\phi}.$$

Integrating, $$\log \frac{1+\sin\phi}{1-\sin\phi} = 2\omega t,$$

no constant being added because $\phi = 0$ when $t = 0$.

From this equation we have

$$\frac{1+\sin\phi}{1-\sin\phi} = \epsilon^{2\omega t}, \quad \sin\phi = \frac{\epsilon^{\omega t} - \epsilon^{-\omega t}}{\epsilon^{\omega t} + \epsilon^{-\omega t}} = \frac{\epsilon^{\psi} - \epsilon^{-\psi}}{\epsilon^{\psi} + \epsilon^{-\psi}},$$

where ψ is equal to $\angle AOx$.

When $t = \infty$, $\sin\phi = 1$, and therefore $\phi = \frac{\pi}{2}$, a value towards which ϕ indefinitely tends as its limit. Thus it appears that after an infinite time the particle will arrive at the point O.

Again, since $r = 2a\cos\phi$, we may readily get

$$r = \frac{4a}{\epsilon^{\omega t} + \epsilon^{-\omega t}} = \frac{4a}{\epsilon^{\psi} + \epsilon^{-\psi}},$$

which gives the distance of the particle from O at any time during the motion.

From the above equations we may obtain for the pressure on the circle corresponding to any position of the particle,

$$R = 2\omega^2 a \cos\phi\,(3\cos\phi - 2).$$

(4) P and Q (fig. 164) are two particles connected together by an inflexible rod PQ without weight; P is capable of moving along a smooth horizontal groove Ox, and Q may move any where upon a smooth horizontal plane passing through the groove; having given the initial circumstances of the particles, to determine their motions at any time after the commencement of the motion.

Let T be the tension of the rod at any time t; θ the inclination of the rod to the line xO; let $ON = x'$, $QN = y'$, where O is an assigned point in Ox, and QN at right angles to ON; $OP = x$; $\omega =$ the initial angular velocity of Q about P, $\beta =$ the initial velocity of P, $\alpha =$ the initial value of θ; let m, m', be the masses of P, Q; a the length of the rod.

For the motion of P there is

$$m\frac{d^2x}{dt^2} = -T\cos\theta \text{.......................} (1);$$

and, for the motion of Q,

$$m'\frac{d^2x'}{dt^2} = T\cos\theta \text{..........................} (2),$$

$$m'\frac{d^2y'}{dt^2} = -T\sin\theta \text{.......................} (3).$$

Adding together the equations (1) and (2),

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0 \text{.....................} (4).$$

Multiplying (1) by $\sin\theta$, (3) by $\cos\theta$, and subtracting the latter of the resulting equations from the former,

$$m\sin\theta\frac{d^2x}{dt^2} - m'\cos\theta\frac{d^2y'}{dt^2} = 0 \text{..............} (5).$$

Again, it is evident that

$$x' = x - a\cos\theta \text{.........................} (6),$$

$$y' = a\sin\theta \text{............................} (7).$$

From (4) and (6) we have

$$(m+m')\frac{d^2x}{dt^2} - m'a\frac{d^2}{dt^2}\cos\theta = 0,$$

and, from (5) and (7),

$$m\sin\theta\frac{d^2x}{dt^2} - m'a\cos\theta\frac{d^2}{dt^2}\sin\theta = 0:$$

eliminating $\dfrac{d^2x}{dt^2}$ between the two last equations,

$$(m+m')\cos\theta\frac{d^2}{dt^2}\sin\theta - m\sin\theta\frac{d^2}{dt^2}\cos\theta = 0;$$

multiplying by $2\dfrac{d\theta}{dt}$, and integrating,

$$(m+m')\left(\frac{d}{dt}\sin\theta\right)^2 + m\left(\frac{d}{dt}\cos\theta\right)^2 = C,$$

$$(m+m'\cos^2\theta)\frac{d\theta^2}{dt^2} = C;$$

but, initially, $\theta = \alpha$, $\frac{d\theta}{dt} = \omega$; hence

$$(m+m'\cos^2\alpha)\,\omega^2 = C,$$

and therefore $$\frac{d\theta^2}{dt^2} = \omega^2\frac{m+m'\cos^2\alpha}{m+m'\cos^2\theta} \quad\text{.................. (8).}$$

Again, integrating (4), we get

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = C,$$

and therefore, by (6),

$$(m+m')\frac{dx}{dt} + m'a\sin\theta\frac{d\theta}{dt} = C;$$

but, at the commencement of the motion, $\frac{dx}{dt} = \beta$, $\theta = \alpha$, $\frac{d\theta}{dt} = \omega$;

hence $$(m+m')\beta + m'a\omega\sin\alpha = C,$$

and therefore $$\frac{dx}{dt} = \beta + \frac{m'a\omega\sin\alpha}{m+m'} - \frac{m'a\sin\theta}{m+m'}\frac{d\theta}{dt};$$

whence, by (8),

$$\frac{dx}{dt} = \beta + \frac{m'a\omega\sin\alpha}{m+m'} - \frac{m'a\omega\sin\theta}{m+m'}\left(\frac{m+m'\cos^2\alpha}{m+m'\cos^2\theta}\right)^{\frac{1}{2}} \quad\text{........ (9).}$$

The equations (8) and (9) give us the velocity of P along Ox, and the angular velocity of Q about P, for any assignable inclination of the rod to the line Ox. If between these two equations we eliminate dt, we shall obtain a differential equation to the path of Q in x and θ.

From (8) we have

$$t = \frac{\omega}{(m+m'\cos^2\alpha)^{\frac{1}{2}}}\int_\alpha^\theta (m+m'\cos^2\theta)^{\frac{1}{2}}\,d\theta,$$

an elliptic transcendent for the determination of t for any value of θ.

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1736, p. 10.

(5) A string is completely coiled round the circumference of a circular lamina, and has a particle attached to one extremity, which is free, the other extremity being fixed to the lamina: every particle of the lamina repels the free particle with a force varying inversely as the distance; to find the velocity of the particle at any time after its departure from the circumference of the lamina.

Let a denote the radius of the lamina, r the distance of the particle from its centre at any time, and f the initial repulsive force experienced by the particle. Then, as may be ascertained by the performance of the appropriate integrations, the repulsive force on the particle at any time from the centre of the lamina will be $\frac{af}{r}$. Hence the particle may be considered as moving along a curve which is the locus of the free extremity of the string acted on by a central repulsive force $\frac{af}{r}$; and therefore, by the formula (D) of Section (I.),

$$v^2 = C + 2\int \frac{af}{r}\,dr$$

$$= C + af \log r^2 = C + af \log (\rho^2 + a^2),$$

if $\rho =$ the length of the string set free.

But, initially, $v = 0$, $\rho = 0$; hence

$$0 = C + af \log a^2,$$

and therefore $$v^2 = af \log \frac{\rho^2 + a^2}{a^2}.$$

Let θ denote the angle subtending the arc of the circumference of the lamina from which the string has been uncoiled; then $\rho = a\theta$, and we have

$$v^2 = af \log (1 + \theta^2).$$

(6) Two particles, connected together by a rigid rod without weight, are projected along a smooth horizontal plane; to determine their motion.

Let the plane of co-ordinates coincide with the plane of the motion. Let m, n, be the resolved parts of the initial velocity of the centre of gravity of the two particles parallel to the axes of x, y, and let a, b, be its initial co-ordinates. Let ω be the initial angular velocity of the rod, θ its inclination to the axis of x at the end of the time t, and ϵ at the beginning of the motion. Then the position of the centre of gravity is given at any time t by the equations

$$x = mt + a, \qquad y = nt + b;$$

and the inclination of the rod to the axis of x, by the equation

$$\theta = \omega t + \epsilon.$$

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1736, p. 7. Euler; *Act. Acad. Petrop.* 1780, P. 1; *Opuscula, De motu corporum flexibilium*, Tom. III. p. 91.

(7) A spherical particle moves within a smooth tube which revolves about one extremity with a uniform angular velocity in a vertical plane, the capacity of the tube being just sufficiently great for the reception of the particle; to determine the motion of the particle.

Let Ox, (fig. 161), which is horizontal, be the initial position of the tube, and P the position of the particle in the tube after a time t. Let ω denote the angular velocity of the tube about O, θ the inclination of OP to Ox, and let $OP = r$. Then, supposing the initial velocity of the particle to be zero, and that $r = a$ initially, the value of r at any time t is given by the equation

$$r = \frac{g}{2\omega^2}\sin\omega t + \frac{2a\omega^2 - g}{4\omega^2}\epsilon^{\omega t} + \frac{2a\omega^2 + g}{4\omega^2}\epsilon^{-\omega t},$$

and the polar equation to the path of the particle will result from the substitution of θ for ωt in this equation. When t becomes very great, the polar equation becomes

$$r = \frac{2a\omega^2 - g}{4\omega^2}\epsilon^{\theta},$$

which is the equation to an equiangular spiral.

The solution of this problem was attempted by M. Le Barbier, in the *Annales de Gergonne*, Tom. XIX. p. 285, who omitted to take into consideration the centrifugal force, an oversight which entirely vitiated his results. The correct solution was given in Tom. XX. by Ampère.

(8) A material particle P (fig. 165) is fixed to one end of a rigid rod PQ without weight lying upon a smooth horizontal plane. The end Q is constrained to move with a uniform velocity in the circumference of a circle ABQ; to find the velocity of the increase of the angle PQR, O being the centre of the circle, and OQR a straight line.

If $PQ = h$, $OQ = a$, $\angle PQR = \psi$ at any time t, $\alpha =$ the initial value of ψ, $\omega =$ the angular velocity of OQ, $\beta =$ the initial value of $\frac{d\psi}{dt}$, then

$$h\left(\frac{d\psi^2}{dt^2} - \beta^2\right) = 2a\omega^2 (\cos\psi - \cos\alpha).$$

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1736, p. 14.

(9) QBA (fig. 166) is a circle on a horizontal plane, and QP a string touching it at the point Q; P is a particle attached to the end of the string. Supposing the particle P to be projected at right angles to QP with a given velocity so as to cause QP to be gradually wrapped about the circumference QBA; to find the velocity of the particle at any time during the motion, and the time which will elapse before the particle reaches the circumference.

Let β be the velocity of projection, v the velocity at any time during the motion, b the length of the string PQ, a the radius of the circle, T the time required. Then

$$v = \beta, \qquad T = \frac{b^2}{2a\beta}.$$

(10) A circular horizontal lamina of matter ABC, (fig. 167), every particle of which attracts with a force varying inversely as the distance, is made to revolve with a uniform angular velocity round an axis through its centre O at right angles to its plane,

the motion taking place in the direction of the arrows; to find the equation to the groove Aa which must be carved in the circular lamina that it may be described freely by a particle subject to the attraction of the lamina; the initial position of the particle being a point A in the circumference of the circle, and its initial velocity being zero.

Let P be any point in the groove; let $OP = r$, $OA = a$, $\angle POA = \theta$, $\omega =$ the angular velocity of the lamina about O, and $f =$ the attraction of the lamina on a particle in its circumference. Then the equation to the groove Aa will be

$$r = a \cos\left\{\left(\frac{f}{\omega^2 a}\right)^{\frac{1}{2}} \theta\right\}.$$

(11) Two small equal bodies A, B, connected together by a rigid line, are placed in a narrow rectilinear tube, in which they can move without friction; the tube is then made to revolve with a uniform angular velocity round a vertical axis which passes through a point C of the tube, this point C lying initially between A and B at a distance a from A and b from B; to find the time of A's arriving at C, and the tension of the rigid line at any time, a being considered less than b.

If ω denote the angular velocity of the tube, m the mass of each particle, t the required time, and T the tension; then

$$t = \frac{1}{\omega} \log \frac{b^{\frac{1}{2}} + a^{\frac{1}{2}}}{b^{\frac{1}{2}} - a^{\frac{1}{2}}}, \quad T = \tfrac{1}{2} m\omega^2 (a + b).$$

(12) A particle is drawn up an indefinitely thin cycloidal tube, the axis of the cycloid being vertical, by means of an equal particle, to which the former particle is attached by a thread passing over a pulley at the highest point of the arc; to find the time of ascending to the highest point.

If T represent the required time, and t the time of a semi-oscillation in the cycloid,

$$T = 2^{\frac{1}{2}} t.$$

(13) A heavy particle having been placed at a point in a straight line within a horizontal plane of indefinite length,

round which as an axis the plane is then made to revolve downwards with a uniform angular velocity; to find what time will elapse before the particle leaves the plane.

If ω be the angular velocity and t the required time, then

$$4 \cos \omega t = \epsilon^{\omega t} + \epsilon^{-\omega t}.$$

This problem was proposed in the *Lady's Diary* for the year 1778, by the celebrated Landen, by whom a solution was given, which is singularly defective, not only in consequence of his neglecting the consideration of centrifugal force, but also from his erroneously supposing the horizontal velocity of the particle to be equal to its velocity along the plane, multiplied by the cosine of the plane's inclination to the horizon. See *Diarian Repository* p. 512, where a correct solution is given by the Editors of the *Repository*, together with Landen's.

SECT. 6. *Constrained Motion of a Particle in Resisting Media.*

(1) A particle descends down a straight line AB, (fig. 168) inclined at an angle α to the vertical, in a medium of uniform density, in which the resistance varies as the velocity; to determine the velocity and the space at the end of any time.

Let P be the position of the particle at the end of any time t, v its velocity; let $AP = x$, and $k =$ the resistance for a unit of velocity. Then, since the resolved part of the force of gravity along AB is at every point $g \cos \alpha$, we have for the motion of P,

$$\frac{dv}{dt} = g \cos \alpha - kv,$$

$$\frac{dv}{g \cos \alpha - kv} = dt.$$

Integrating, we have

$$C - \frac{1}{k} \log (g \cos \alpha - kv) = t;$$

but $v = 0$ when $t = 0$, and therefore

$$C - \frac{1}{k} \log (g \cos \alpha) = 0,$$

hence $$\frac{1}{k}\log\frac{g\cos\alpha - kv}{g\cos\alpha} = -t,$$

$$\frac{g\cos\alpha - kv}{g\cos\alpha} = \epsilon^{-kt},$$

$$v = \frac{g\cos\alpha}{k}(1-\epsilon^{-kt}),$$

which gives the velocity for any value of t.

Again, since $dx = vdt$, we have

$$x = \frac{g\cos\alpha}{k}\int(1-\epsilon^{-kt})\,dt$$

$$= C + \frac{g\cos\alpha}{k^2}(kt+\epsilon^{-kt}):$$

but, A being considered the initial position of the particle,

$$0 = C + \frac{g\cos\alpha}{k^2};$$

hence $$x = \frac{g\cos\alpha}{k^2}(kt - 1 + \epsilon^{-kt}),$$

which gives the position of the particle at any time.

Euler; *Mechan.* Tom. II. p. 244.

(2) A particle descends from rest by the action of gravity from a point E, (fig. 169), down the arc EA of a cycloid BAB', of which the axis AC is vertical; the motion takes place in a medium of uniform density, where the resistance is partly constant and partly proportional to the square of the velocity; to find the velocity of the particle when it arrives at the point A, and to determine at what point in its descent its velocity is a maximum.

Let $AM = x$, $AP = s$, $AE = c$, $v =$ the velocity at P; then, h and k being constant quantities, the equation of motion along the curve will be

$$v\frac{dv}{ds} = -g\frac{dx}{ds} + h + \frac{v^2}{k};$$

hence $$d.v^2 - \frac{2ds}{k}v^2 = -2g\,dx + 2h\,ds:$$

but, by the nature of the cycloid, if $\frac{1}{2}a$ be the radius of the generating circle, $dx = \frac{s}{a}\,ds$; hence

$$d\,.\,v^2 - \frac{2ds}{k}\,v^2 = -\frac{2g}{a}\,sds + 2hds.$$

Multiplying both sides of the equation by $\epsilon^{-\frac{2s}{k}}$, we have

$$d\left(v^2\epsilon^{-\frac{2s}{k}}\right) = 2h\epsilon^{-\frac{2s}{k}}ds - \frac{2g}{a}\,\epsilon^{-\frac{2s}{k}}\,sds.$$

Integrating, we have

$$v^2\epsilon^{-\frac{2s}{k}} = C - hk\epsilon^{-\frac{2s}{k}} - \frac{2g}{a}\int\epsilon^{-\frac{2s}{k}}\,sds;$$

but
$$\int\epsilon^{-\frac{2s}{k}}\,sds = -\tfrac{1}{2}k\,\epsilon^{-\frac{2s}{k}}s + \tfrac{1}{2}k\int\epsilon^{-\frac{2s}{k}}\,ds$$

$$= -\tfrac{1}{2}k\epsilon^{-\frac{2s}{k}}s - \tfrac{1}{4}k^2\epsilon^{-\frac{2s}{k}};$$

hence
$$v^2\epsilon^{-\frac{2s}{k}} = C - hk\epsilon^{-\frac{2s}{k}} + \frac{1}{a}\left(gks + \tfrac{1}{2}gk^2\right)\epsilon^{-\frac{2s}{k}},$$

$$= C + \frac{1}{a}\left(gks + \tfrac{1}{2}gk^2 - ahk\right)\epsilon^{-\frac{2s}{k}}.$$

But, initially, $v = 0$, $s = c$; hence

$$0 = C + \frac{1}{a}\left(gkc + \tfrac{1}{2}gk^2 - ahk\right)\epsilon^{-\frac{2c}{k}}.$$

Let v_1 be the value of v when $s = 0$; then

$$v_1^2 = C + \frac{1}{a}\left(\tfrac{1}{2}gk^2 - ahk\right);$$

and therefore
$$v_1^2 = \frac{k}{a}\left(\tfrac{1}{2}gk - ah\right) - \frac{k}{a}\left(gc + \tfrac{1}{2}gk - ah\right)\epsilon^{-\frac{2c}{k}}.$$

Again, $v^2\epsilon^{-\frac{2s}{k}} = \frac{k}{a}\left(gs + \tfrac{1}{2}gk - ah\right)\epsilon^{-\frac{2s}{k}} - \frac{k}{a}\left(gc + \tfrac{1}{2}gk - ah\right)\epsilon^{-\frac{2c}{k}}$,

$$v^2 = \frac{k}{a}\left(gs + \tfrac{1}{2}gk - ah\right) - \frac{k}{a}\left(gc + \tfrac{1}{2}gk - ah\right)\epsilon^{\frac{2}{k}(s-c)}.$$

When v is a maximum,

$$0=\frac{gk}{a}-\frac{2}{a}(gc+\tfrac{1}{2}gk-ah)\,\epsilon^{\frac{2}{k}(s-c)},$$

$$\epsilon^{\frac{2}{k}(s-c)}=\frac{\frac{1}{2}gk}{gc+\frac{1}{2}gk-ah},$$

$$s=c+\frac{k}{2}\log\frac{\frac{1}{2}gk}{gc+\frac{1}{2}gk-ah}=c-\frac{k}{2}\log\frac{gc+\frac{1}{2}gk-ah}{\frac{1}{2}gk},$$

which gives the position of the particle for a maximum velocity.

Euler; *Mechan.* Tom. II. p. 292.

(3) From a given point O, (fig. 170), an infinite number of straight lines OP are drawn in a vertical plane; to determine the nature of the curve APD, such that a particle descending down any line OP may always acquire the same velocity on arriving at P, the medium in which the motion takes place being uniform, and its resistance varying as any power of the velocity.

Let β be the velocity at P, v at any point p in OP; let $OP=r$, $Op=z$; $\angle POx=\theta$, Ox being vertical; draw PM horizontally, and let $OM=x$; then, k being the resistance for a unit of velocity, and m the index of its power,

$$v\frac{dv}{dz}=g\cos\theta-kv^m,$$

$$dz=\frac{vdv}{g\cos\theta-kv^m}.$$

Integrating, we have

$$r=\int_0^\beta\frac{vdv}{g\cos\theta-kv^m}=\int_0^\beta\frac{\beta d\beta}{g\cos\theta-k\beta^m}.$$

But $x=r\cos\theta$; hence

$$x=\int_0^\beta\frac{\beta d\beta}{g-k\beta^m\sec\theta}\quad\text{.......................}(1)$$

$$=\frac{\frac{1}{2}\beta^2}{g-k\beta^m\sec\theta}-\tfrac{1}{2}mk\sec\theta\int_0^\beta\frac{\beta^{m+1}d\beta}{(g-k\beta^m\sec\theta)^2}\quad\text{..........}(2).$$

But, β being a constant quantity while x and θ vary, we have, from (1),

$$dx = kd(\sec\theta) . \int_0^\beta \frac{\beta^{m+1} d\beta}{(g - k\beta^m \sec\theta)^2};$$

and therefore, by (2),

$$2x = \frac{\beta^2}{g - k\beta^m \sec\theta} - \frac{m \sec\theta \, dx}{d\sec\theta};$$

or, putting $\frac{r}{x}$ for $\sec\theta$,

$$m\frac{r}{x} dx + 2xd\left(\frac{r}{x}\right) = \frac{\beta^2 d\left(\frac{r}{x}\right)}{g - k\frac{r}{x}\beta^m},$$

and therefore $\quad (m-2)\, r dx + 2x\, dr = \beta^2 \dfrac{xdr - rdx}{gx - k\beta^m r};$

which is the differential equation to the curve in x and r.

Euler; *Mechan.* Tom. II. p. 246.

(4) To find the tautochrone in a medium the resistance of which varies as the square of the velocity, the particle being acted on by gravity.

Let O (fig. 171) be the point to which the particle is always to descend in the same time, AO being the tautochrone. Take Oy horizontal as the axis of y, Ox vertical as the axis of x. Let $OM = x$, $OP = s$; $v =$ the velocity of the particle at P, and $k =$ the resistance of the medium for a unit of velocity.

The equation for the motion along the curve will be

$$vdv = -gdx + kv^2 ds:$$

multiplying by $2\epsilon^{-2ks}$, we have

$$d(v^2\epsilon^{-2ks}) = -2g\epsilon^{-2ks}\, dx.$$

Integrating, we obtain

$$v^2\epsilon^{-2ks} = C - 2g\int \epsilon^{-2ks}\, dx.$$

Suppose the velocity of the particle on its arrival at O to be that due to an altitude h in vacuum; then

$$2gh = C - 2g\int^0 \epsilon^{-2kx}\,dx;$$

hence $$v^2\epsilon^{-2ks} = 2g\left\{h - \int_0^x \epsilon^{-2kx}\,dx\right\} \text{..................(1)},$$

and therefore, v being equal to $-\dfrac{ds}{dt}$ at any time t,

$$dt = -\frac{1}{(2g)^{\frac{1}{2}}}\,\frac{\epsilon^{-ks}\,ds}{(h-u)^{\frac{1}{2}}},$$

where $$u = \int_0^x \epsilon^{-2kx}\,dx \text{..............................(2)}.$$

Now, s being some function of x and therefore of u, we may assume $\epsilon^{-ks}\,ds = \phi(u)\,du$, and thus

$$dt = -\frac{1}{(2g)^{\frac{1}{2}}}\,\frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}, \quad t = -\frac{1}{(2g)^{\frac{1}{2}}}\int\frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}.$$

But $t = 0$ when $v = 0$, and therefore, by (1) and (2), when $u = h$; and it will denote the whole time of descent to O from the beginning of the motion when $x = 0$, and therefore, by (2), when $u = 0$; hence the whole time of the descent is equal to

$$\frac{1}{(2g)^{\frac{1}{2}}}\int_0^h \frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}} \text{..........................(3)},$$

a result which, for the condition of tautochronism, must evidently be independent of h. From this it is plain that

$$\int\frac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}$$

must be of no dimensions in u and h together, and that consequently its differential $\dfrac{\phi(u)\,du}{(h-u)^{\frac{1}{2}}}$ must be of no dimensions in u, h, du; hence, since $\phi(u)$ evidently does not involve h, we must have

$$\phi(u) = \frac{a}{u^{\frac{1}{2}}},$$

where a is some constant quantity. Hence, putting for $\phi(u)$ its value,

$$\epsilon^{-ks}\,ds = a\,\frac{du}{u^{\frac{1}{2}}}.$$

Integrating, we have

$$C - \frac{1}{k}\epsilon^{-ks} = 2au^{\frac{1}{2}}.$$

But, by (2), when s and therefore x is equal to zero, $u = 0$;

hence $$C - \frac{1}{k} = 0;$$

hence $$\frac{1}{k}(1 - \epsilon^{-ks}) = 2au^{\frac{1}{2}}, \quad \frac{1}{k^2}(1 - \epsilon^{-ks})^2 = 4a^2u,$$

$$\frac{1}{k}(1 - \epsilon^{-ks})\,\epsilon^{-ks} = 2a^2\frac{du}{ds} = 2a^2\epsilon^{-2ks}\frac{dx}{ds}, \text{ by (2)};$$

and therefore the equation to the tautochrone will be

$$2a^2k\frac{dx}{ds} = \epsilon^{ks} - 1.$$

Since $\phi(u) = \frac{a}{u^{\frac{1}{2}}}$, we have, from (3), if τ denote the whole time of descent,

$$\tau = \frac{\pi a}{(2g)^{\frac{1}{2}}}, \quad a^2 = \frac{2g\tau^2}{\pi^2};$$

and therefore the equation to the tautochrone for the time τ will be

$$4gk\tau^2\frac{dx}{ds} = \pi^2(\epsilon^{ks} - 1)\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Euler; *Comment. Petrop.* 1729; *Mechan.* Tom. II. p. 392. John Bernoulli; *Mém. de l'Acad. des Sciences de Paris*, 1730; *Opera*, Tom. III. p. 173. See also the *Cambridge Mathematical Journal*, Vol. II. p. 153, where Mr. Leslie Ellis has reduced the solution of this problem to that of the Tautochrone in vacuum.

(5) A particle oscillates in an inverted cycloid, of which the axis is vertical, in a uniform medium where the resistance varies as the velocity; having given the first arc of descent, to find the whole space described by the particle before the motion ceases.

Let c denote the first arc of descent, k the resistance when the velocity is unity, a the radius of the generating circle; then the whole space will be equal to

$$c\,\frac{\epsilon^{\frac{\phi}{2}}+1}{\epsilon^{\frac{\phi}{2}}-1}, \quad \text{where } \phi = \frac{k\pi}{\left(\frac{g}{a}-k^2\right)^{\frac{1}{2}}}.$$

(6) An inelastic particle descends down the sides of a plane equilateral and equiangular polygon in a vertical plane, the medium in which the motion takes place being uniform and its resistance varying as the square of the velocity; to determine the velocity of the particle when it has arrived at the end of any side of the polygon, the side down which the particle first descends being vertical.

Let $\pi - \alpha$ be the magnitude of each of the angles of the polygon, l the length of each of its sides, kv^2 the resistance for a velocity v; v_x the velocity at the end of the x^{th} side. Then

$$v_x = (\cos\alpha)^x\,\epsilon^{-klx}\left\{A^2 + \frac{g\,(\epsilon^{2kl}-1)}{k\,(\cos\alpha)^2}\cdot\frac{M\cos x\alpha + N\sin x\alpha}{M^2+N^2}\;\frac{\epsilon^{2klx}}{(\cos\alpha)^{2x}}\right\}^{\frac{1}{2}},$$

where $$M = \frac{\epsilon^{2kl}-\cos\alpha}{\cos\alpha}, \qquad N = \frac{\sin\alpha}{(\cos\alpha)^2}\,\epsilon^{2kl};$$

and A is an arbitrary constant which will easily be determined if we know the value of v_x for any value of x.

Bordoni; *Memorie della Societa Italiana*, 1816, p. 173.

(7) From a given point O (fig. 170) an infinite number of straight lines OP are drawn in a vertical plane; to determine the nature of the curve APD, so that a particle descending down any line OP may always acquire the same velocity on arriving at P; the motion taking place in a medium of uniform density where the resistance varies as the the square of the velocity.

Let OA be vertical and be represented by a; let $OP = r$, $OM = x$, $k =$ the resistance for a unit of velocity; then the equation to the curve will be

$$x = r\,\frac{\epsilon^{2kr}}{\epsilon^{2ka}}\,\frac{\epsilon^{2ka}-1}{\epsilon^{2kr}-1}.$$

Euler; *Mechan.* Tom. II. p. 251.

(8) From a given point O (fig. 172) an infinite number of straight lines OP are drawn in a vertical plane; to determine the nature of the curve OPA so that a particle may descend through all its chords OP in the same time; the motion taking place in a medium of uniform density where the resistance varies as the square of the velocity.

Let OA be vertical and be equal to a; let $OP = r$, $\angle AOP = \theta$; then the polar equation to the curve OPA will be

$$(\cos\theta)^{\frac{1}{2}} = \frac{\log\{\epsilon^{kr} + (\epsilon^{2kr} - 1)^{\frac{1}{2}}\}}{\log\{\epsilon^{ka} + (\epsilon^{2ka} - 1)^{\frac{1}{2}}\}}.$$

Euler; *Mechan.* Tom. II. p. 256.

(9) A particle, acted on by gravity, is ascending the curve MNA, (fig. 173), in a medium where the resistance varies as the square of the velocity; to find the nature of the curve that the velocity of the particle at any point N may be the same as that which it would acquire by falling in the same medium down a vertical line IN, the length of which is equal to the arc AN of the curve measured from a fixed point A.

Draw through A a vertical line AB, and let fall NQ at right angles to AB; let $AQ = x$, $AN = s$, and $k =$ the resistance of the medium for a unit of velocity. Then the differential equation to the curve will be

$$k(x + s) = 1 - \epsilon^{-2ks}.$$

This problem was proposed to Clairaut on his journey to Lapland, by Klingstierna, Professor of Mathematics at Upsal, at which place he called on his way; Klingstierna's construction, together with his own solution, was published by Clairaut in the *Mémoires de l'Académie des Sciences de Paris*, 1740, p. 254.

CHAPTER V.

MOMENT OF INERTIA.

The Moment of Inertia of a body with regard to any axis, is the sum of all the products resulting from the multiplication of each element of the mass by the square of its distance from the axis. If M denote the whole mass of the body, the Moment of Inertia may be represented by the expression Mk^2, where k is a line called the Radius of Gyration. The term Moment of Inertia was first made use of by Euler. "Ratio hujus denominationis ex similitudine motus progressivi est desumpta: quemadmodum enim in motu progressivo, si a vi secundum suam directionem sollicitante acceleretur, est incrementum celeritatis ut vis sollicitans divisa per massam seu inertiam; ita in motu gyratorio, quoniam loco ipsius vis sollicitantis ejus momentum considerari oportet, eam expressionem $\int r^2 dM$, quæ loco inertiæ in calculum ingreditur, *momentum inertiæ* appellemus, ut incrementum celeritatis angularis simili modo proportionale fiat momento vis sollicitantis diviso per momentum inertiæ[1]."

Sect. 1. *A Material Curve revolving about an Axis within its own Plane.*

(1) To find the moment of inertia and radius of gyration of a circular arc about a radius through its vertex.

Let HAK (fig. 174) be the circular arc, A its vertex, C the centre of the circle. Take any point P in the arc; draw PM at right angles to the radius CA; join HK, intersecting CA in E. Join CH, CK, CP. Let $PM=y$, $CA=a$, arc $AP=s$, HE $=c=KE$, $\angle ACH=\alpha=\angle ACK$, $\angle PCA=\theta$. Then, the density of the arc and the indefinitely small area of the section of it

[1] Euler; *Theoria Motus Corporum Solidorum*, p. 167.

made by a plane through C, at right angles to its own plane, being represented respectively by ρ and κ, we shall have

$$Mk^2 = \kappa\rho \int y^2 ds$$

$$= \kappa\rho \int_{-\alpha}^{+\alpha} a^2 \sin^2\theta \,.\, a d\theta$$

$$= \tfrac{1}{2}\kappa\rho a^3 \int_{-\alpha}^{+\alpha} (1 - \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\kappa\rho a^3 (2\alpha - \sin 2\alpha):$$

but $$M = \kappa\rho \,.\, 2a\alpha:$$

hence $$k^2 = \tfrac{1}{2}a^2 \left(1 - \frac{\sin 2\alpha}{2\alpha}\right)$$

$$= \tfrac{1}{2}a^2 - \frac{a \sin\alpha \,.\, a\cos\alpha}{2\alpha}$$

$$= \tfrac{1}{2}a^2 - \frac{c\,(a^2 - c^2)^{\frac{1}{2}}}{2\sin^{-1}\dfrac{c}{a}}.$$

If the arc be a semicircle, $c = a$, and, if a circle, $c = 0$; in both cases $k^2 = \frac{1}{2}a^2$.

(2) To find the radius of gyration of a material straight line OB, (fig. 175), about an axis OA, to which it is inclined at a given angle, the density at any point of OB varying as some power of its distance from O.

Take any point P in OB; draw PM at right angles to OA; let $PM = y$, $OP = s$, $OB = l$, $\angle AOB = \alpha$, $\rho =$ the density at P: then $\rho = \mu s^n$, where μ is a constant quantity. Hence

$$Mk^2 = \kappa \int \rho y^2 ds = \kappa\mu \sin^2\alpha \int_0^l s^{n+2}\, ds = \frac{\kappa\mu \sin^2\alpha\, l^{n+3}}{n+3}.$$

Also, $$M = \kappa \int \rho\, ds = \kappa\mu \int_0^l s^n\, ds = \frac{\kappa\mu\, l^{n+1}}{n+1}.$$

Hence we get $$k^2 = \frac{n+1}{n+3}\, l^2 \sin^2\alpha.$$

If the density be invariable, $n = 0$, and $k^2 = \frac{1}{3} l^2 \sin^2\alpha$.

SECT. 2. *Material Line revolving about an Axis at Right Angles to its own Plane.*

(1) To find the radius of gyration of a straight line AB, (fig. 176) about an axis through D at right angles to the plane ADB.

Let C be the middle point of the line; join CD. Let $AC = a = BC$, $CD = b$; $k =$ the radius of gyration about the axis through D, and $k' =$ that about an axis parallel to this through C. Then

$$k^2 = k'^2 + b^2.$$

But, $2a\rho\kappa$ being the mass of AB,

$$2a\rho\kappa k'^2 = \rho\kappa \int_{-a}^{+a} s^2 ds = \frac{2\rho\kappa}{3} a^3,$$

$$k'^2 = \tfrac{1}{3} a^2.$$

Hence $$k^2 = \tfrac{1}{3} a^2 + b^2.$$

(2) To find the radius of gyration of a circular arc about an axis perpendicular to its plane through its centre of gravity.

Let k be the radius of gyration about the required axis, k' about an axis parallel to this through the centre of the circle, and h the distance between the centre of gravity of the arc and the centre of the circle. Then

$$k'^2 = k^2 + h^2.$$

But, r denoting the radius of the circle, c the chord, and a the length of the arc,

$$k'^2 = r^2, \quad h^2 = \frac{c^2 r^2}{a^2}.$$

Hence, for the required radius of gyration,

$$k^2 = \frac{r^2}{a^2} (a^2 - c^2).$$

(3) To find the radius of gyration of a circular arc about an axis perpendicular to its plane through its vertex.

If $r =$ the radius of the circle, $a =$ the length and $c =$ the chord of the arc,

$$k^2 = 2r^2 \left(1 - \frac{c}{a}\right).$$

(4) If the density of a straight rod AB vary as the n^{th} power of the distance from one end A, and k, k', be the radii of gyration of the rod round axes at right angles to its length through A and B respectively; to compare the values of k and k', and to ascertain the value of n so that k may be equal to $6k'$.

$$\frac{k^2}{k'^2} = \frac{(n+1)(n+2)}{2}; \qquad n = 7, \text{ or } = -10.$$

SECT. 3. *A Plane Area revolving about an Axis within or parallel to the Plane.*

(1) To find the radius of gyration of an elliptic area, of uniform thickness and density, about its principal axes.

Let ρ represent the uniform density of the area, and τ its indefinitely small thickness; then, x, y, denoting the co-ordinates of any point of the curve referred to a, b, as axes of co-ordinates, we have for the moment of inertia, about the axis a, of a quadrant of the ellipse,

$$Mk^2 = \rho\tau \int_0^b y^2 . x dy$$

$$= \frac{\rho\tau a}{b} \int_0^b y^2 (b^2 - y^2)^{\frac{1}{2}} dy:$$

but

$$\int_0^b y^2 (b^2 - y^2)^{\frac{1}{2}} dy = \tfrac{1}{3} \int_0^b (b^2 - y^2)^{\frac{3}{2}} dy$$

$$= \tfrac{1}{4} b^2 \int_0^b (b^2 - y^2)^{\frac{1}{2}} dy$$

$$= \tfrac{1}{16} \pi b^4.$$

Hence the moment of inertia of the whole ellipse will be equal to

$$4Mk^2 = \tfrac{1}{4} \pi\rho\tau a b^3;$$

but

$$4M = \pi\rho\tau ab;$$

hence

$$k^2 = \tfrac{1}{4} b^2.$$

If k' denote the radius of gyration about the axis b, we shall have, by similar reasoning,

$$k'^2 = \tfrac{1}{4} a^2.$$

(2) To find the radius of gyration of a circular area revolving about a straight line parallel to its plane, at a distance c from its centre.

If a be the radius of the circle, and k the required radius of gyration,

$$k^2 = \tfrac{1}{4}a^2 + c^2.$$

(3) To find the radius of gyration of an isosceles triangle about a perpendicular let fall from its vertex upon its base.

If $2b =$ the length of the base,

$$k^2 = \tfrac{1}{6}b^2.$$

SECT. 4. *Plane Area about a Perpendicular Axis.*

(1) To find the radius of gyration of a triangular lamina ABC, (fig. 177), about an axis through A at right angles to its plane.

Take two points P, p, indefinitely near to each other in the side AB, and draw PM, pm, parallel to BC. Take P', p', in PM, pm, and construct the indefinitely small parallelogram $P'p'$, two of the sides of which are parallel to AC. Let $AM = x$, $PM = y$, $P'M = y'$, $Am = x + dx$, $p'm = y' + dy'$, $\angle ACB = C$; let a, b, c, be the three sides of the triangle.

Then, Mk^2 denoting the moment of inertia about A, we have, τ denoting the indefinitely small thickness, and ρ the density of the lamina,

$$\begin{aligned} Mk^2 &= \int_0^b \int_0^y (x^2 + y'^2 - 2xy' \cos C)\, \rho\tau \sin C\, dx\, dy' \\ &= \rho\tau \sin C \int_0^b (x^2 y + \tfrac{1}{3}y^3 - xy^2 \cos C)\, dx \\ &= \rho\tau \sin C \int_0^b \left(\frac{a}{b}x^3 + \frac{1}{3}\frac{a^3}{b^3}x^3 - \frac{a^2}{b^2}x^3 \cos C\right) dx \\ &= \tfrac{1}{12} \cdot \tfrac{1}{2}\rho\tau\, ab \sin C\,(6b^2 + 2a^2 - 6ab \cos C) \\ &= \tfrac{1}{12} M \{6b^2 + 2a^2 - 3(a^2 + b^2 - c^2)\}, \end{aligned}$$

and therefore

$$k^2 = \tfrac{1}{12}(3b^2 + 3c^2 - a^2).$$

(2) To find the radius of gyration of a triangular lamina ABC about a perpendicular through its centre of gravity G.

Let AG, BG, CG, be represented by α, β, γ; and BC, CA, AB, by a, b, c. Then, M denoting the mass of the whole triangle ABC, $\frac{1}{3}M$ will be the mass of each of the triangles BGC, CGA, AGB. Hence, by the preceding problem, the moment of inertia of these three triangles respectively about the axis through G will be

$$\tfrac{1}{36}M(3\beta^2+3\gamma^2-a^2),$$
$$\tfrac{1}{36}M(3\gamma^2+3\alpha^2-b^2),$$
$$\tfrac{1}{36}M(3\alpha^2+3\beta^2-c^2);$$

and therefore the moment of inertia of the whole triangle about G will be equal to

$$\tfrac{1}{36}M\{6(\alpha^2+\beta^2+\gamma^2)-(a^2+b^2+c^2)\};$$

or, by a property of the centre of gravity of a triangle, to

$$\tfrac{1}{36}M\{2(a^2+b^2+c^2)-(a^2+b^2+c^2)\}$$
$$=\tfrac{1}{36}M(a^2+b^2+c^2).$$

Hence the radius of gyration will be equal to

$$\tfrac{1}{36}(a^2+b^2+c^2).$$

Euler; *Theoria Motus Corporum Solidorum*, cap. VI. Prob. 32. Cor. 1.

(3) To find the radius of gyration of an elliptic area about a perpendicular axis through its centre.

If M be the mass of the area, the moment of inertia about the two axes of the ellipse will be

$$\tfrac{1}{4}Mb^2, \qquad \tfrac{1}{4}Ma^2.$$

But the moment of inertia of a plane area, about any perpendicular axis, is equal to the sum of the moments of inertia about any two lines, at right angles to each other in the plane area, passing through the point in which the axis meets the area. Hence, in the present problem, the moment of inertia about the proposed axis is equal to

$$\tfrac{1}{4}M(a^2+b^2),$$

and the radius of gyration $=\frac{1}{4}(a^2+b^2)$.

(4) To find the radius of gyration of an annulus about a perpendicular axis through the centre.

Let r be the distance of any point of the annular area from the centre of the circle, θ the angular co-ordinate, ρ the density, and τ the indefinitely small thickness of the area; then a, b, being the radii of the two concentric circles,

$$Mk^2 = \int_0^{2\pi}\int_a^b r^2 . \rho\tau r d\theta dr$$

$$= \tfrac{1}{4}\rho\tau\int_0^{2\pi}(b^4 - a^4)\, d\theta = \tfrac{1}{2}\pi\rho\tau\,(b^4 - a^4).$$

But $$M = \int_0^{2\pi}\int_a^b \rho\tau r d\theta dr = \tfrac{1}{2}\rho\tau\int_0^{2\pi}(b^2 - a^2)\, d\theta$$

$$= \pi\rho\tau\,(b^2 - a^2);$$

hence $$k^2 = \tfrac{1}{2}(a^2 + b^2).$$

(5) A disk of any form, the mass, density, and thickness of which are given, revolves round an axis perpendicular to its plane: to determine the form of the disk and the position of the axis in order that the moment of inertia may be a minimum.

Let mk^2 denote its moment of inertia, m denoting its mass: then, ρ being its density, and τ its thickness, x and y being polar co-ordinates,

$$mk^2 = \rho\tau\int_0^y\int_0^{2\pi} y\, dx\, dy\, .\, y^2 = \tfrac{1}{4}\rho\tau\int_0^{2\pi} y^4\, dx,$$

$$m = \rho\tau\int_0^y\int_0^{2\pi} y\, dx\, dy \quad = \tfrac{1}{2}\rho\tau\int_0^{2\pi} y^2\, dx.$$

Put $v = \int y^4\, dx, \qquad v = \int y^2\, dx$:

then, a being some constant,

$$u - av = \int (y^4 - ay^2)\, dx,$$

$$V = y^4 - ay^2.$$

Hence, by the formula of the Calculus of Variations,

$$N - \frac{d\,(P)}{dx} + \frac{d^2\,(Q)}{dx^2} - \&c. = 0,$$

we have, since P, Q, R, &c., are all zero,

$$4y^3 - 2ay = 0, \qquad y^2 = \tfrac{1}{2}a,$$

which shews that the disk is circular, its centre being in the axis of rotation.

Also $$m = \tfrac{1}{2}\rho\tau \int_0^{2\pi} y^2 dx = \tfrac{1}{4}\rho\tau a \int_0^{2\pi} dx = \tfrac{1}{2}\pi\rho\tau a,$$

$$a = \frac{2m}{\pi\rho\tau},$$

and therefore y, the radius, is equal to $\left(\dfrac{m}{\pi\rho\tau}\right)^{\frac{1}{2}}$.

(6) To find the radius of gyration of a parallelogram about an axis perpendicular to it through its centre of gravity.

If $2a$, $2b$, be the lengths of two adjoining sides of the parallelogram, then, whatever be the angle of their inclination,

$$k^2 = \tfrac{1}{3}(a^2 + b^2).$$

Euler; *Theoria Motus Corp. Solid.* Cap. VI. Prob. 35.

(7) To find the radius of gyration of a regular polygon about an axis perpendicular to it through the centre. If n be the number of sides, and c the length of each,

$$k^2 = \tfrac{1}{24}c^2 \frac{2 + \cos\dfrac{2\pi}{n}}{1 - \cos\dfrac{2\pi}{n}}.$$

(8) To find the radius of gyration of a portion of a parabola bounded by a double ordinate to the axis about a perpendicular line through its vertex.

If x, y, represent the extreme co-ordinates of the portion,

$$k^2 = \tfrac{3}{7}x^2 + \tfrac{1}{5}y^2.$$

SECT. 5. *Plane Area about an Oblique Axis.*

Having given the greatest of the moments of inertia of any plane figure about the three principal axes, which have the

same origin, to find the moment of inertia about an axis passing through the same origin and equally inclined to the three principal axes.

Let A, B, C, be the moments of inertia about the three principal axes: one of these will evidently be at right angles to the plane area; we will suppose C to correspond to this. Then, μ being the moment of inertia about the other axis,

$$\mu = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$$
$$= (A + B + C)\cos^2\alpha,$$

since $\alpha = \beta = \gamma$, by the supposition.

But we know that $A + B = C$: hence

$$\mu = 2C\cos^2\alpha,$$

C being evidently greater than either A or B.

But, since $\qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$

and $\alpha = \beta = \gamma$, we see that $\cos^2\alpha = \frac{1}{3}$: hence,

$$\mu = \tfrac{2}{3}C.$$

SECT. 6. *Symmetrical Solid about its Axis.*

(1) To find the radius of gyration of a homogeneous sphere about a diameter.

Let x, $x + dx$, be the distances of the circular faces of a thin circular slice of the sphere, at right angles to the diameter, from the centre, and let y be the radius of the section. Then, ρ denoting the density of the sphere, the moment of inertia of this slice about the diameter will be equal to

$$\tfrac{1}{2}\pi\rho y^4 dx;$$

and therefore the moment of inertia of the whole sphere, a being its radius, will be equal to

$$\tfrac{1}{2}\pi\rho\int_{-a}^{+a} y^4 dx = \tfrac{1}{2}\pi\rho\int_{-a}^{+a}(a^2 - x^2)^2\,dx = \tfrac{8}{15}\pi\rho a^5.$$

But the mass of the sphere is equal to $\frac{4}{3}\pi\rho a^3$; hence

$$k^2 = \tfrac{2}{5}a^2.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 198.

(2) To find the radius of gyration of a right cone about its axis.

If a denote the radius of the base of the cone,

$$k^2 = \tfrac{3}{10} a^2.$$

Euler; *Ib.* p. 197.

(3) To find the radius of gyration of a hollow sphere about a diameter.

If a, b, be the external and internal radii,

$$k^2 = \tfrac{2}{5} \frac{a^5 - b^5}{a^3 - b^3}.$$

Euler; *Ib.* p. 203.

(4) To find the radius of gyration of a solid cylinder about its axis.

If a denote the radius of the cylinder,

$$k^2 = \tfrac{1}{2} a^2.$$

Euler; *Ib.* p. 200.

(5) To find the moment of inertia of a sphere about a diameter, the density varying as the n^{th} power of the distance from the centre.

If $\mu =$ the density at a unit of distance from the centre, and $a =$ the radius, the moment of inertia is equal to

$$\frac{8\pi\mu}{3\,(n+5)} \cdot a^{n+5}.$$

(6) To find the radii of gyration of an ellipsoid about its axes.

If h, k, l, be the radii of gyration about the axes $2a$, $2b$, $2c$, respectively,

$$h^2 = \frac{b^2 + c^2}{5}, \quad k^2 = \frac{c^2 + a^2}{5}, \quad l^2 = \frac{a^2 + b^2}{5}.$$

SECT. 7. *Moment of Inertia of a Solid not Symmetrical with respect to the Axis of Gyration.*

(1) To find the radius of gyration of a solid cylinder about an axis perpendicular to its own through its middle point.

Let x be the distance of any thin circular slice of the cylinder from the middle point of its axis; dx the thickness of the slice; ρ the density of the cylinder, b its radius, and $2a$ its length. Then, the moment of inertia of the slice about any diameter being equal to

$$\tfrac{1}{4}\pi\rho b^4 dx,$$

its moment of inertia about the axis of gyration in the present problem will be equal to

$$\pi\rho b^2 dx \,.\, (x^2 + \tfrac{1}{4}b^2).$$

Hence, Mk^2 denoting the moment of inertia of the whole cylinder about the proposed axis, we have

$$\begin{aligned} Mk^2 &= \pi\rho b^2 \int_{-a}^{+a} (x^2 + \tfrac{1}{4}b^2)\, dx \\ &= \pi\rho b^2 \left(\tfrac{2}{3}a^3 + \tfrac{1}{2}ab^2\right) \\ &= 2\,\pi\rho a b^2 \left(\tfrac{1}{3}a^2 + \tfrac{1}{4}b^2\right); \end{aligned}$$

and therefore, M being equal to $2\pi\rho ab^2$, we have

$$k^2 = \tfrac{1}{3}a^2 + \tfrac{1}{4}b^2.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 196.

(2) To determine the moment of inertia of an ellipsoid about the diagonal of the inscribed parallelepiped of maximum volume.

If α, β, γ, be the angles made by the diagonal with the axes of x, y, z,

$$\cos^2\alpha = \frac{x^2}{r^2}, \quad \cos^2\beta = \frac{y^2}{r^2}, \quad \cos^2\gamma = \frac{z^2}{r^2},$$

where $$r^2 = x^2 + y^2 + z^2,$$

x, y, z, being the co-ordinates of an extremity of the diagonal.

Also, M being the mass of the ellipsoid, the moments of inertia about the axes of x, y, z, are, as is proved in ordinary works on Dynamics,

$$\frac{1}{5}M(b^2 + c^2), \qquad \frac{1}{5}M(c^2 + a^2), \qquad \frac{1}{5}M(a^2 + b^2).$$

Hence the required moment of inertia is equal to

$$\frac{1}{5}M\left\{(b^2 + c^2)\frac{x^2}{r^2} + (c^2 + a^2)\frac{y^2}{r^2} + (a^2 + b^2)\frac{z^2}{r^2}\right\}.$$

But, when the parallelepiped is a maximum, we may easily ascertain that

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}:$$

hence the moment of inertia becomes

$$\frac{2}{5} M . \frac{b^2c^2 + c^2a^2 + a^2b^2}{a^2 + b^2 + c^2}.$$

(3) To find the moment of inertia of a right cone, the base of which is a segment of a sphere, about an axis, through its vertex, perpendicular to its axis.

Let Oz (fig. 178) be the axis of the cone; Ox, Oy, two lines through the vertex O at right angles to Oz; PN a perpendicular from P, any point in the cone, upon the plane xOy: join ON, and draw NM at right angles to Oy: join PO, PM. Let $m =$ an element of the cone at the point P, $Mk^2 =$ the required moment of inertia about Oy.

Let $a =$ the radius of the sphere, $PO = r$, $2\beta =$ the vertical angle of the cone, $\angle POz = \theta$, $\angle NOx = \phi$.

Then $$Mk^2 = \Sigma\,(m \,.\, PM^2).$$

But $$m = r d\theta\, dr\, r \sin\theta\, d\phi,$$

and $$PM^2 = r^2 \cos^2\theta + r^2 \sin^2\theta \cos^2\phi:$$

hence $$Mk^2 = \int_0^a \int_0^{2\pi} \int_0^{\beta} r^4 (\sin\theta \cos^2\theta + \sin^3\theta \cos^2\phi)\, d\theta\, d\phi\, dr$$

$$= \tfrac{1}{5} a^5 \int_0^{2\pi} \int_0^{\beta} (\sin\theta \cos^2\theta + \sin^3\theta \cos^2\phi)\, d\theta\, d\phi$$

$$= \tfrac{1}{5} \pi a^5 \int_0^{\beta} (2 \sin\theta \cos^2\theta + \sin^3\theta)\, d\theta$$

$$= \tfrac{1}{5} \pi a^5 \,.\, {}_0^{\beta}\!\left(-\tfrac{1}{3} \cos^3\theta - \cos\theta\right)$$

$$= \tfrac{1}{15} \pi a^5 (4 - 3 \cos\beta - \cos^3\beta).$$

If $\beta = \pi$, or the cone become a sphere,

$$Mk^2 = \tfrac{8}{15} \pi a^5; \quad \text{but } M = \tfrac{4}{3} \pi a^3: \quad \text{hence } k^2 = \tfrac{2}{5} a^2.$$

(4) To find the radius of gyration of a right cone about an axis through its vertex at right angles to its geometrical axis.

If $a =$ the altitude of the cone, and $c =$ the radius of the base,

$$k^2 = \tfrac{3}{20}(4a^2 + c^2).$$

(5) To find the radius of gyration of a right cone about an axis at right angles to the axis of the cone and passing through its centre of gravity.

If a be the altitude of the cone, and c the radius of its base; then

$$k^2 = \tfrac{3}{80}(a^2 + 4c^2).$$

Euler; *Ib.* p. 197.

(6) To find the radius of gyration of a double convex lens about its axis, and about a diameter to the circle in which its two spherical surfaces intersect; the two surfaces having equal radii.

If $a =$ the semi-axis of the lens, $b =$ the radius of the circular intersection of the two surfaces; $k =$ the radius of gyration of the lens about its axis, and k' about a diameter of the circle; we shall have

$$k^2 = \tfrac{1}{10}\frac{a^4 + 5a^2b^2 + 10b^4}{a^2 + 3b^2}, \quad k'^2 = \tfrac{1}{20}\frac{7a^4 + 15a^2b^2 + 10b^4}{a^2 + 3b^2}.$$

Euler; *Ib.* p. 201.

(7) The height of a right cone being h, and the radius of its base r, to find the moment of inertia of the cone about a straight line joining its vertex and a point in the circumference of its base.

The required moment of inertia is equal to

$$\tfrac{1}{20}\pi \cdot \frac{hr^4}{h^2 + r^2} \cdot (6h^2 + r^2).$$

CHAPTER VI.

D'ALEMBERT'S PRINCIPLE.

A GENERAL method for the determination of the motion of a material system, acted on by any forces, was laid down by D'Alembert in his *Traité de Dynamique*, published in the year 1743[1], from which we have extracted the following passage in exposition of the Principle[2].

"*Problême Général.*

"Soit donné un systême de corps disposés les uns par rapport aux autres d'une manière quelconque; et supposons qu'on imprime à chacun de ces corps un mouvement particulier, qu'il ne puisse suivre à cause de l'action des autres corps; trouver le mouvement que chaque corps doit prendre.

"*Solution.*

"Soient A, B, C, &c. les corps qui composent le systême, et supposons qu'on leur ait imprimé les mouvemens a, b, c, etc. qu'ils soient forcés, à cause de leur action mutuelle, de changer dans les mouvemens a, b, c, etc. Il est clair qu'on peut regarder le mouvement a imprimé au corps A comme composé du mouvement a, qu'il a pris, et d'un autre mouvement α; qu'on peut de même regarder les mouvemens b, c, etc. comme composés des mouvemens b, β; c, κ; etc. d'où il s'ensuit que le mouvement des corps A, B, C, etc. entr'eux auroit été le même, si au lieu de leur donner les impulsions a, b, c, etc. on leur eût donné à-la-fois les doubles impulsions a, α; b, β; c, κ, etc. Or par la supposition, les corps A, B, C, etc. ont pris d'eux-mêmes les mouve-

[1] See also his *Recherches sur la Précession des Equinoxes*, p. 35, published in 1749.

[2] D'Alembert's Principle was first enunciated by him in a memoir which he read before the Academy of Sciences at the end of the year 1742.

mens a, b, c, etc. donc les mouvemens α, β, κ, etc. doivent être tels qu'ils ne derangent rien dans les mouvemens a, b, c, etc. c'est-à-dire que, si les corps n'avoient reçu que les mouvemens α, β, κ, etc. ces mouvemens auroient dû se détruire mutuellement, et le systême demeurer en repos.

"De là résulte le principe suivant, pour trouver le mouvement de plusieurs corps qui agissent les uns sur les autres. *Décomposez les mouvemens a, b, c, etc. imprimés à chaque corps, chacun en deux autres* a, α; b, β; c, κ; *etc. qui soient tels, que si l'on n'eût imprimé aux corps que les mouvemens* a, b, c, *etc. ils eussent pu conserver ces mouvemens sans se nuire réciproquement; et que si on ne leur eût imprimé que les mouvemens* α, β, κ, *etc. le systême fût demeuré en repos; il est clair que* a, b, c, *etc. seront les mouvemens que ces corps prendront en vertu de leur action. Ce qu'il falloit trouver.*"

The idea of the general method developed by D'Alembert for the determination of the motion of material systems, had occurred somewhat earlier to Fontaine, as we are informed in the *Table des Mémoires*, prefixed to his *Traité de Calcul Différentiel et Integral*[1], having been communicated by him to the Academy of Sciences in the year 1739, and subsequently to several mathematicians. His views, however, on this subject were not made public till long after the appearance of the *Traité de Dynamique*; and in all probability D'Alembert, who did not become a member of the Academy before the year 1741, was not aware of Fontaine's generalization. D'Alembert, however, was the first to shew the wonderful fertility of the Principle by applying it to the solution of a great variety of difficult problems, among which may be mentioned that of the Precession of the Equinoxes, which had been inadequately attempted by Newton, and of which D'Alembert was the first to obtain a complete solution.

The earliest step towards the discovery of D'Alembert's Principle is to be met with in a memoir by James Bernoulli in the *Acta Eruditorum*, 1686, Jul. p. 356, entitled "Narratio Controversiæ inter Dn. Hugenium et Abbatem Catelanum agitatæ de Centro Oscillationis quæ loco animadversionis esse poterit in

[1] *Mémoires de l'Académie des Sciences de Paris*, 1770.

Responsionem Dn. Catelani, num. 27. Ephem. Gallic. anni 1684, insertam." Let m, m', denote two equal bodies attached to an inflexible straight line which is capable of motion in a vertical plane about one extremity which is fixed; let r, r', denote the distances of m, m', respectively, from the fixed extremity; v, v', their velocities for any position of the inflexible line in its descent from an assigned position; u, u', the velocities which they would have acquired by descending down the same arcs unconnectedly. Then, in consequence of the connection of the bodies, a velocity $u - v$ will be lost by m and a velocity $v' - u'$ gained by m' in their descent. Bernoulli proposes it to the consideration of mathematicians whether, according to the statical relation of two forces in equilibrium on a lever, the proportion $u - v : v' - u' :: r' : r$ be an accurate expression of the circumstances of the motion. This idea of Bernoulli's, although not free from error, contains however the first germ of the Principle of reducing the determination of the motions of material systems to the solution of statical problems. L'Hôpital, in a letter addressed to Huyghens[1], correctly observed that instead of considering the velocities acquired in a finite time, he should have considered the infinitesimal velocities acquired in an instant of time, and have compared them with those which gravity tends to impress upon the bodies during the same instant. He takes a complex pendulum, consisting of any two bodies attached to an inflexible straight line, and considers equilibrium to subsist between the quantities of motion lost and gained by these bodies in any instant of time, that is, between the differences of the quantities of motion which the bodies really acquire in this instant, and those which gravity tends to impress on them. He applies this Principle, which agrees with the general Principle of D'Alembert, to the determination of the Centre of Oscillation of a pendulum consisting of two bodies attached to an inflexible straight line oscillating about one extremity. He then extends his theory to a greater number of bodies in a straight line, and determines their Centre of Oscillation on the supposition, the truth of which is not however sufficiently obvious without demonstration, that any two of them

[1] *Histoire des Ouvrages des Sçavans*, 1690, *Juin*, p. 440.

may be collected at their particular Centre of Oscillation. On the publication of L'Hôpital's letter, James Bernoulli[1] reverted to the subject of the Centre of Oscillation, and at length succeeded in obtaining a direct and rigorous solution of the problem in the case where all the bodies are in one line, by the application of the principle laid down by L'Hôpital. Bernoulli[2] afterwards extended his method to the general case of the oscillations of bodies of any figure.

An ingenious investigation of the Centre of Oscillation, a problem from the beginning intimately connected with the development of D'Alembert's Principle, was shortly afterwards given by Brook Taylor[3] and John Bernoulli[4], between whom arose an angry controversy respecting priority of discovery[5]; the method given by these mathematicians, although depending upon the statical principles of the lever, did not however involve, in an explicit form, L'Hôpital's Principle of Equilibrium. Finally, Hermann[6] determined the Centre of Oscillation by the principle of the statical equivalence of the *solicitations of gravity*, and the *vicarious solicitations* applied in opposite directions, or, as it is expressed by modern mathematicians, by the equilibrium subsisting between the impressed forces of gravity and the effective forces applied in opposite directions; a method of investigation virtually coincident with that given by James Bernoulli. The idea of L'Hôpital became still more general in the hands of Euler[7], in a memoir on the determination of the oscillations of flexible strings printed in the year 1740. From the above historical sketch it will be easily seen that in the enunciation of a general Principle of Motion, Fontaine and D'Alembert had little more to do than to express in general language what had been distinctly conceived in the prosecution of particular researches by L'Hôpital, James and John Bernoulli, Brook Taylor,

[1] *Acta Erudit. Lips.* 1691. *Jul.* p. 317. *Opera*, Tom. I. p. 460.

[2] *Mémoires de l'Académie des Sciences de Paris*, 1703, 1704.

[3] *Philosophical Transactions*, 1714, *May*. *Methodus Incrementorum*.

[4] *Acta Erudit. Lips.* 1714, *Jun.* p. 257; *Mém. Acad. Par.* 1714, p. 208. *Opera*, Tom. II. p. 168.

[5] *Act. Erudit. Lips.* 1716, 1718, 1719, 1721, 1722.

[6] *Phoronomia*; Lib. I. cap. 5.

[7] *Comment. Petrop.* Tom. VII.

Hermann, and Euler. For additional information on the historical development of D'Alembert's Principle, the reader is referred to Lagrange's *Mécanique Analytique*, Seconde Partie, Section 1; Montucla's *Histoire des Mathématiques*, part. V. liv. 3, part. IV. liv. 7; and Whewell's *History of the Inductive Sciences*, Vol. II.

In modern treatises on Mechanics, D'Alembert's Principle is expressed under one or other of the following forms:

(1) When any material system is in motion under the action of any forces, the moving forces lost by the different molecules of the system must be in equilibrium.

(2) If the effective moving forces of the several particles of a system be applied to them in directions opposite to those in which they act; they will, conjointly with the impressed moving forces, constitute a system of forces statically disposed.

The former of these enunciations it will be seen is substantially the same as that given by D'Alembert, while the latter is a generalization of the idea developed by Hermann in his investigations on the particular problem of the Centre of Oscillation.

SECT. 1. *Motion of a single Particle*[1].

The object of this section is to apply D'Alembert's Principle to the exemplification of a general method for the determination of the motion of a particle within tubes and between contiguous surfaces, of which either the position, or the form, or both, are made to vary according to any assigned law whatever, the particle being acted on by given forces. Several of the problems of this section have been solved by particular methods in Chapter IV.

I. We will commence with the consideration of the motion of a particle along a tube, and, for the sake of perfect generality, we will suppose the tube to be one of double curvature. The tube is considered in all cases to be indefinitely narrow and perfectly smooth, and every section at right angles to its axis to be circular.

[1] The substance of this Section was published in the *Cambridge Mathematical Journal*, Vol. III. p. 49.

Let the particle be referred to three fixed rectangular axes, and let x, y, z, be its co-ordinates at any time t; let x, y, z, become $x+\delta x$, $y+\delta y$, $z+\delta z$, when t becomes $t+\delta t$; δt, and consequently δx, δy, δz, being considered to be indefinitely small. Then the effective accelerating forces on the particle parallel to the three fixed axes will be, at the time t,

$$\frac{\delta^2 x}{\delta t^2},\ \frac{\delta^2 y}{\delta t^2},\ \frac{\delta^2 z}{\delta t^2}.$$

Also, let X, Y, Z, represent the impressed accelerating forces on the particle resolved parallel to the axes of x, y, z; and let $x+dx$, $y+dy$, $z+dz$, be the co-ordinates of a point in the tube very near to the point x, y, z, which the particle occupies at the time t. Then, observing that the action of the tube on the particle is always at right angles to its axis at every point and therefore, at the time t, to the line joining the two points x, y, z, and $x+dx$, $y+dy$, $z+dz$, we have, by D'Alembert's Principle, combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2}-X\right)dx+\left(\frac{\delta^2 y}{\delta t^2}-Y\right)dy+\left(\frac{\delta^2 z}{\delta t^2}-Z\right)dz=0 \ldots\ldots (\text{A}).$$

Again, since the form and position of the tube are supposed to vary according to some assigned law, it is clear that when t is known the equations to the tube must be known; hence it is evident that, in addition to the equation (A), we shall have, from the particular conditions of each individual problem, a number of equations equivalent to two of the form

$$\phi(x, y, z, t)=0, \qquad \chi(x, y, z, t)=0 \ldots\ldots\ldots (\text{B}),$$

where ϕ and χ are symbols of functionality depending upon the law of the variations of the form and position of the tube.

The three equations (A) and (B) involve the four quantities x, y, z, t, and therefore, in any particular case, if the difficulty of the analytical processes be not insuperable, we may ascertain x, y, z, each of them in terms of t; in which consists the complete solution of the problem.

If the tube remain during the whole of the motion within one plane, then, the plane of x, y, being so chosen as to coincide

with this plane, the three equations (A) and (B) will evidently be reduced to the two

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy = 0 \text{ .... (C)},$$

$$\phi(x, y, t) = 0 \text{ .... (D)}.$$

We proceed to illustrate the general formulæ of the motion by the discussion of a few problems.

(1) A rectilinear tube revolves with a uniform angular velocity about one extremity in a horizontal plane; to find the motion of a particle within the tube.

Let ω be the constant angular velocity; r the distance of the particle at any time t from the fixed extremity of the tube; then, the plane of x, y, being taken horizontal, and the origin of co-ordinates at the fixed extremity of the tube, we shall have, supposing the tube initially to coincide with the axis of x,

$$x = r \cos \omega t \text{ .... (1)},$$

$$y = r \sin \omega t \text{ .... (2)}.$$

From (1) we have

$$dx = dr \cos \omega t,$$

and, from (2),

$$dy = dr \sin \omega t.$$

Again, from (1) we have

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t;$$

and, from (2),

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t.$$

Substituting in the general formula (C) the values which we have obtained for dx, dy, $\frac{\delta^2 x}{\delta t^2}$, $\frac{\delta^2 y}{\delta t^2}$, we have, since $X = 0$, $Y = 0$,

$$\cos\omega t\left(\frac{\delta^2 r}{\delta t^2}\cos\omega t - 2\omega\frac{\delta r}{\delta t}\sin\omega t - \omega^2 r\cos\omega t\right)$$

$$+\sin\omega t\left(\frac{\delta^2 r}{\delta t^2}\sin\omega t + 2\omega\frac{\delta r}{\delta t}\cos\omega t - \omega^2 r\sin\omega t\right) = 0;$$

and therefore

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0;$$

the integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t}.$$

Let $r = a$ when $t = 0$; then

$$a = C + C';$$

also let $\frac{\delta r}{\delta t} = \beta$ when $t = 0$; then

$$\beta = C\omega - C'\omega;$$

from the two equations for the determination of C and C', we have

$$C = \frac{a\omega + \beta}{2\omega}, \qquad C' = \frac{a\omega - \beta}{2\omega};$$

hence, for the motion of the particle along the tube,

$$2\omega r = (a\omega + \beta)\,\epsilon^{\omega t} + (a\omega - \beta)\,\epsilon^{-\omega t}.$$

This problem, which is the earliest problem of the motion of a particle subject to the constraint of a curve moving according to a prescribed law, is due to John Bernoulli[1]. A solution of this problem is given also by Clairaut[2], to whom it had probably been proposed by Bernoulli.

(2) Supposing the tube to revolve in a vertical instead of a horizontal plane, we shall have, by the same process, the axis of y being now taken vertical, observing that $X = 0$, $Y = -g$, if the time be reckoned from the moment of coincidence of the tube with the axis of x which is horizontal,

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = -g\sin\omega t.$$

[1] *Opera*, Tom. IV. p. 248.

[2] *Mémoires de l'Académie des Sciences de Paris*, 1742, p 10.

The integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t;$$

and, if we determine the constants from the conditions that r, $\frac{\delta r}{\delta t}$, shall have initially values a, β, we shall get for the motion along the tube,

$$2\omega r = \left(a\omega + \beta - \frac{g}{2\omega}\right)\epsilon^{\omega t} + \left(a\omega - \beta + \frac{g}{2\omega}\right)\epsilon^{-\omega t} + \frac{g}{\omega}\sin\omega t.$$

This problem, which had been erroneously attempted by Barbier in the *Annales de Gergonne*, Tom. XIX., was correctly solved, in the following volume, by Ampère. In the *Cambridge Mathematical Journal*, Vol. III. p. 42, a solution is given by Professor Booth, who has discussed at length the more interesting cases of the motion.

(3) Suppose the tube to revolve in a horizontal plane about a fixed extremity with such an angular velocity, that the tangent of its angle of inclination to the axis of x is proportional to the time.

The equation to the tube at any time t will be

$$y = mtx \text{..............................} (1),$$

where m is some constant quantity; hence

$$dy = mt\,dx,$$

and therefore, from (C), since $X = 0$ and $Y = 0$,

$$\frac{\delta^2 x}{\delta t^2} + mt\frac{\delta^2 y}{\delta t^2} = 0 \text{........................} (2).$$

But from (1) we have

$$\frac{\delta y}{\delta t} = mt\frac{\delta x}{\delta t} + mx,$$

$$\frac{\delta^2 y}{\delta t^2} = mt\frac{\delta^2 x}{\delta t^2} + 2m\frac{\delta x}{\delta t};$$

hence, from (2),

$$(1 + m^2t^2)\frac{\delta^2 x}{\delta t^2} + 2m^2t\frac{\delta x}{\delta t} = 0,$$

$$\frac{\frac{\delta^2 x}{\delta t^2}}{\frac{\delta x}{\delta t}} + \frac{2m^2 t}{1 + m^2 t^2} = 0.$$

Integrating, we have

$$\log \frac{\delta x}{\delta t} + \log (1 + m^2 t^2) = \log C,$$

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = C.$$

Let β be the initial value of $\frac{\delta x}{\delta t}$, which will be the velocity of projection along the tube; then $C = \beta$, and therefore

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = \beta, \qquad \delta x = \frac{\beta \delta t}{1 + m^2 t^2};$$

integrating, we get

$$x + C = \frac{\beta}{m} \tan^{-1} (mt),$$

Let $x = a$ when $t = 0$; then $a + C = 0$, and therefore

$$x = a + \frac{\beta}{m} \tan^{-1} (mt),$$

and consequently, from (1),

$$y = amt + \beta t \tan^{-1} (mt).$$

If θ be the inclination of the tube to the axis of x at any time, and r be the distance of the particle from the fixed extremity,

$$r = \frac{am + \beta\theta}{m \cos \theta}.$$

(4) A circular tube is constrained to move in a horizontal plane with a uniform angular velocity about a fixed point in its circumference; to determine the motion of a particle within the tube, which is placed initially in the extremity of the diameter passing through the fixed point.

Let the fixed point be taken as the origin of co-ordinates, and let the axis of x coincide with the initial position of the diameter through this point; let ω be the angular velocity of the revolution of the circle, a the radius; also let θ be the angle, at any time t, between the distance of the particle and of the extremity of the diameter through the origin from the centre of the circle.

Then it will be easily seen that

$$x = a \cos \omega t + a \cos (\omega t - \theta) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$y = a \sin \omega t + a \sin (\omega t - \theta) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

From (1) we have

$$dx = a d\theta \sin (\omega t - \theta),$$

and, from (2), $\qquad dy = - a d\theta \cos (\omega t - \theta).$

Hence, from (C), observing that $X = 0$ and $Y = 0$,

$$\sin (\omega t - \theta) \frac{\delta^2 x}{\delta t^2} - \cos (\omega t - \theta) \frac{\delta^2 y}{\delta t^2} = 0 \;\ldots\ldots\ldots\; (3).$$

Again, from (1),

$$\frac{\delta x}{\delta t} = - a\omega \sin \omega t + a \left(\frac{\delta \theta}{\delta t} - \omega\right) \sin (\omega t - \theta),$$

$$\frac{\delta^2 x}{\delta t^2} = - a\omega^2 \cos \omega t - a \left(\frac{\delta \theta}{\delta t} - \omega\right)^2 \cos (\omega t - \theta) + a \frac{\delta^2 \theta}{\delta t^2} \sin (\omega t - \theta);$$

and, from (2),

$$\frac{\delta y}{\delta t} = a\omega \cos \omega t - a \left(\frac{\delta \theta}{\delta t} - \omega\right) \cos (\omega t - \theta),$$

$$\frac{\delta^2 y}{\delta t^2} = - a\omega^2 \sin \omega t - a \left(\frac{\delta \theta}{\delta t} - \omega\right)^2 \sin (\omega t - \theta) - a \frac{\delta^2 \theta}{\delta t^2} \cos (\omega t - \theta);$$

and therefore, by (3),

$$a\omega^2 \{\sin \omega t \cos (\omega t - \theta) - \cos \omega t \sin (\omega t - \theta)\} + a \frac{\delta^2 \theta}{\delta t^2} = 0,$$

$$\omega^2 \sin \theta + \frac{\delta^2 \theta}{\delta t^2} = 0:$$

multiplying by $2 \frac{\delta \theta}{\delta t}$, and integrating,

$$\frac{\delta \theta^2}{\delta t^2} = C + 2\omega^2 \cos \theta.$$

But, the absolute velocity of the particle being initially zero, it is clear that 2ω will be the initial value of $\frac{\delta \theta}{\delta t}$; and therefore, θ being initially zero, we have

$$4\omega^2 = C + 2\omega^2, \qquad C = 2\omega^2,$$

and therefore

$$\frac{\delta \theta^2}{\delta t^2} = 2\omega^2 (1 + \cos \theta) = 4\omega^2 \cos^2 \frac{\theta}{2}, \quad \frac{\delta \theta}{\delta t} = 2\omega \cos \frac{\theta}{2},$$

$$\frac{\cos\frac{\theta}{2}\,\delta\theta}{\cos^2\frac{\theta}{2}} = 2\omega\delta t, \qquad \frac{\delta\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}} = \omega\delta t.$$

Integrating, we have

$$\log\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} = 2\omega t + C;$$

but $\theta = 0$ when $t = 0$; hence $C = 0$, and we have

$$\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} = \epsilon^{2\omega t},$$

and therefore

$$\sin\frac{\theta}{2} = \frac{\epsilon^{\omega t}-\epsilon^{-\omega t}}{\epsilon^{\omega t}+\epsilon^{-\omega t}},$$

which determines the position of the particle within the tube at any time. When $t = \infty$, we have $\sin\frac{\theta}{2} = 1$, and therefore $\theta = \pi$, which shews that, after the lapse of an infinite time, the particle will arrive at the point of rotation.

(5) If we pursue the same course as in the solution of the problems (1), (2), (4), we may obtain a convenient formula for the following more general problem: a plane curvilinear tube of any invariable form whatever revolves in its own plane about a fixed point with a uniform angular velocity; to determine the motion of a particle, acted on by any forces, within the tube.

Let ω be the constant angular velocity of the tube about the fixed point; r the distance of the particle at any time from this point; ϕ the angle between the simultaneous directions of r and of a line joining an assigned point of the tube with the fixed point of rotation; ds an element of the length of the tube at the place of the particle, and S the accelerating force on the particle resolved along the element ds; then the equation for the motion of the particle will be

$$r^2\frac{\delta\phi^2}{\delta t^2}+\frac{\delta r^2}{\delta t^2}-\omega^2r^2=2\int S\frac{ds}{d\phi}\,\delta\phi:$$

but since, the form of the tube being invariable, $\delta\phi$, δr, may evidently be replaced by $d\phi$, dr, we have, putting, for the sake of uniformity of notation, dt in place of δt,

$$r^2\frac{d\phi^2}{dt^2}+\frac{dr^2}{dt^2}-\omega^2r^2=2\int S\,ds.$$

If ω be zero, the formula will become

$$r^2\frac{d\phi^2}{dt^2}+\frac{dr^2}{dt^2}=2\int Sds,$$

the well-known formula for the motion of a particle under the action of any forces within an immoveable plane tube.

(6) In the foregoing examples the position of the tube varies with the time; the form however remains invariable. We will now give an example in which the form changes with the time.

A particle is projected with a given velocity within a circular tube, the radius of which increases in proportion to the time while the centre remains stationary; to determine the motion of the particle, the tube being supposed to lie always in a horizontal plane.

The equation to the circle will be

$$x^2+y^2=a^2\,(1+\alpha t)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a and α are some constant quantities; hence

$$x\,dx+y\,dy=0,$$

and therefore, by the general formula (C),

$$y\frac{\delta^2x}{\delta t^2}-x\frac{\delta^2y}{\delta t^2}=0;$$

integrating, we have

$$y\frac{\delta x}{\delta t}-x\frac{\delta y}{\delta t}=C.$$

Let the axis of x be so chosen as to coincide with the initial distance of the particle from the centre, and let β be the initial velocity of the particle along the tube; then $C=-a\beta$, and therefore

$$x\frac{\delta y}{\delta t}-y\frac{\delta x}{\delta t}=a\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

again, from (1), we have

$$x\frac{\delta x}{\delta t}+y\frac{\delta y}{\delta t}=a^2\alpha\,(1+\alpha t)\;\ldots\ldots\ldots\ldots (3);$$

multiplying (2) by y and (3) by x, and subtracting the former result from the latter, we have

$$(x^2+y^2)\frac{\delta x}{\delta t}=a^2\alpha\,(1+\alpha t)\,x-a\beta y,$$

and therefore, by (1),

$$a\,(1+\alpha t)^2\frac{\delta x}{\delta t}=a\alpha\,(1+\alpha t)\,x-\beta\,\{a^2(1+\alpha t)^2-x^2\}^{\frac{1}{2}}.$$

Put $1+\alpha t=\tau$; then

$$a\alpha\tau^2\frac{\delta x}{\delta \tau}=a\alpha\tau x-\beta\,(a^2\tau^2-x^2)^{\frac{1}{2}};$$

again, put $x=m\tau$, and there is

$$a\alpha\tau^2\left(m+\tau\frac{\delta m}{\delta\tau}\right)=a\alpha m\tau^2-\beta\tau\,(a^2-m^2)^{\frac{1}{2}},$$

$$a\alpha\tau^3\frac{\delta m}{\delta\tau}=-\beta\tau\,(a^2-m^2)^{\frac{1}{2}},$$

$$-a\alpha\frac{\delta m}{(a^2-m^2)^{\frac{1}{2}}}=\beta\frac{\delta\tau}{\tau^2};$$

integrating, $$C+a\alpha\cos^{-1}\frac{m}{a}=-\frac{\beta}{\tau},$$

or, putting for m its value,

$$C+a\alpha\cos^{-1}\frac{x}{a\tau}=-\frac{\beta}{\tau},$$

and, putting for τ its value $1+\alpha t$,

$$C+a\alpha\cos^{-1}\frac{x}{a\,(1+\alpha t)}=-\frac{\beta}{1+\alpha t}.$$

Now $x=a$ when $t=0$; hence $C=-\beta$, and therefore

$$a\alpha\cos^{-1}\frac{x}{a\,(1+\alpha t)}=\frac{\alpha\beta t}{1+\alpha t},$$

$$x = a(1 + \alpha t) \cos \frac{\beta t}{a(1 + \alpha t)},$$

and therefore, from (1),

$$y = a(1 + \alpha t) \sin \frac{\beta t}{a(1 + \alpha t)};$$

which give the absolute position of the particle at any assigned time.

II. We proceed now to the consideration of the motion of a particle along a surface from which it is unable to detach itself, while the surface itself changes its position or its form, or both, according to any assigned law. To fix the ideas, we suppose the particle to move between two surfaces indefinitely close together, so as to be expressed by the same equation.

Let x, y, z, be the co-ordinates of the particle at any time t; and let $\delta x, \delta y, \delta z$, be the increments of x, y, z, in an indefinitely small time δt; also let dx, dy, dz, denote the increments of x, y, z, in passing from the point x, y, z, to any point near to it within the surface as it exists at the time t. Also let X, Y, Z, denote the resolved parts of the accelerating forces on the particle at the time t parallel to the axes of x, y, z; then, observing that the action of the surface on the particle is always in the direction of the normal at each point, we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \text{........(A').}$$

Again, since the position and form of the surface vary according to an assigned law, its equation must evidently be known at any given time, and therefore we must have, from the nature of each particular problem, certain conditions between the quantities x, y, z, t, equivalent to a single equation

$$F = f(x, y, z, t) = 0 \text{.......................(B').}$$

Taking the total differential of (B'), we have

$$\frac{dF}{dx} dx + \frac{dF}{dy} dy + \frac{dF}{dz} dz = 0;$$

eliminating dz between this equation and (A'), we get

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz}dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz}dy$$
$$= \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\left(\frac{dF}{dx}dx + \frac{dF}{dy}dy\right);$$

but dx and dy are independent quantities; we have therefore, by equating separately their coefficients on each side of the equation,

$$\left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dy},$$
$$\left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dx} = \left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz},$$

and therefore also

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dy} = \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dx}:$$

any two of these three relations, together with the equation (B'), will give us three equations in x, y, z, t, whence x, y, z, are to be determined in terms of t.

The following example will serve to illustrate the use of these equations. We have taken a case where the form of the surface remains invariable, its position alone being liable to change. The analysis, however, in the solution of problems of the class which we are considering, receives its general character solely in consequence of the presence of t in the equation (B'), and therefore the example which we have chosen is sufficient for the general object we have in view.

A particle descends by the action of gravity down a plane which revolves uniformly about a vertical axis through which it passes; to determine the motion of the particle.

Let the plane of x, y, be taken horizontal, the axis of x coinciding with the initial intersection of the revolving plane with the horizontal plane through the origin, and let the axis of z be taken vertically downwards; then, ω denoting the angular velocity of the plane, its equation at any time t will be

$$F = y\cos\omega t - x\sin\omega t = 0 \ldots\ldots\ldots\ldots\ldots(1);$$

whence $\quad \dfrac{dF}{dx} = -\sin\omega t, \quad \dfrac{dF}{dy} = \cos\omega t, \quad \dfrac{dF}{dz} = 0;$

also, $X=0$, $Y=0$, $Z=g$; and therefore, from either of the first two of the three general relations,

$$\frac{\delta^2 z}{\delta t^2}=g \quad\ldots\ldots\ldots\ldots (2),$$

and, from the third,

$$\frac{\delta^2 x}{\delta t^2}\cos\omega t+\frac{\delta^2 y}{\delta t^2}\sin\omega t=0 \quad\ldots\ldots\ldots\ldots (3).$$

Let r denote the distance of the particle at any time from the axis of z; then

$$x=r\cos\omega t,\quad y=r\sin\omega t,$$

whence
$$\frac{\delta x}{\delta t}=\frac{\delta r}{\delta t}\cos\omega t-\omega r\sin\omega t,$$

$$\frac{\delta^2 x}{\delta t^2}=\frac{\delta^2 r}{\delta t^2}\cos\omega t-2\omega\frac{\delta r}{\delta t}\sin\omega t-\omega^2 r\cos\omega t,$$

$$\frac{\delta y}{\delta t}=\frac{\delta r}{\delta t}\sin\omega t+\omega r\cos\omega t,$$

$$\frac{\delta^2 y}{\delta t^2}=\frac{\delta^2 r}{\delta t^2}\sin\omega t+2\omega\frac{\delta r}{\delta t}\cos\omega t-\omega^2 r\sin\omega t;$$

and therefore, from (3),

$$\frac{\delta^2 r}{\delta t^2}-\omega^2 r=0 \quad\ldots\ldots\ldots\ldots (4).$$

Let the initial values of z, $\frac{\delta z}{\delta t}$, be 0, β; and those of r, $\frac{\delta r}{\delta t}$, be a, α; then, from the equations (2) and (4), after executing obvious operations, we shall obtain

$$z=\tfrac{1}{2}gt^2+\beta t,$$

$$2\omega r=(\omega a+\alpha)\,\epsilon^{\omega t}+(\omega a-\alpha)\,\epsilon^{-\omega t},$$

and
$$\log\frac{(\omega^2 r^2+\alpha^2-\omega^2 a^2)^{\frac{1}{2}}+\omega r}{\alpha+\omega a}=\frac{\omega}{g}\left\{(2gz+\beta^2)^{\frac{1}{2}}-\beta\right\};$$

the first two of these equations give the position of the particle on the revolving plane, and therefore, by virtue of the equation (1), the absolute position of the particle at any time; while the third is the equation to the path which the particle describes on the plane.

SECT. 2. *Systems of Particles.*

(1) Two heavy particles P, P', (fig. 179), are attached to a rigid imponderable rod APP' which is oscillating in a vertical plane about a fixed point in its extremity A; to determine the motion.

Let m, m', be the masses of the two particles; let $AP = a$, $AP' = a'$. Draw AB vertically downwards and let $\angle PAB = \theta$. Let ds, ds', denote the elements of the circular paths described by P, P', in a small time dt, estimated in a direction corresponding to an increase of θ. Then the effective moving forces of the two particles will be $m\dfrac{d^2s}{dt^2}$, $m'\dfrac{d^2s'}{dt^2}$, the moments of which about the point A will be $ma\dfrac{d^2s}{dt^2}$, $m'a'\dfrac{d^2s'}{dt^2}$. Also the moments of the impressed forces will be $-mag\sin\theta$, $-m'a'g\sin\theta$. Hence, for the equilibrium of the impressed forces, and the effective forces applied in directions opposite to their own, we have

$$ma\frac{d^2s}{dt^2} + m'a'\frac{d^2s'}{dt^2} + (ma + m'a')\,g\sin\theta = 0.$$

But $ds = a d\theta$, $ds' = a'd\theta$; hence

$$(ma^2 + m'a'^2)\frac{d^2\theta}{dt^2} + (ma + m'a')\,g\sin\theta = 0;$$

a result which shews that the rod will oscillate isochronously with a perfect pendulum of which the length is

$$\frac{ma^2 + m'a'^2}{ma + m'a'}.$$

(2) Two particles, attached to the extremities of a fine inextensible thread, are placed upon two inclined planes with a common summit; to determine the motion of the particles and the tension of the thread at any time.

Let m, m', be the masses of the particles; α, α', the inclinations of the planes to the horizon; x, x', the distances of the particles from the common summit of the planes at any time. Then the

impressed accelerating forces on the particles m, m', estimated down the two planes, will be $g \sin \alpha$, $g \sin \alpha'$, respectively, and the effective accelerating forces, estimated in the same directions, will be $\frac{d^2x}{dt^2}$, $\frac{d^2x'}{dt^2}$. Hence, for the equilibrium of the impressed forces, and the effective forces applied in directions opposite to their own, we have

$$g\,(m \sin \alpha - m' \sin \alpha') = m\frac{d^2x}{dt^2} - m'\frac{d^2x'}{dt^2} \qquad \text{..........(1).}$$

But, if c denote the length of the thread,

$$x + x' = c, \qquad \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0:$$

hence, from (1),

$$(m + m')\frac{d^2x}{dt^2} = g\,(m \sin \alpha - m' \sin \alpha') \qquad \text{..............(2);}$$

which determines the common acceleration of the two particles estimated in accordance with an increase of x: should the expression for $\frac{d^2x}{dt^2}$ be a negative quantity, x will decrease and x' increase.

If T denote the tension of the thread, we shall have, for the equilibrium of the impressed moving forces T, $mg \sin \alpha$, exerted on the particle m, and the effective moving force $m\frac{d^2x}{dt^2}$ applied in a direction opposite to its own,

$$T = m\left(g \sin \alpha - \frac{d^2x}{dt^2}\right)$$

$$= \frac{mm'g}{m + m'}(\sin \alpha + \sin \alpha'), \text{ by (2)};$$

which gives the value of T, which is therefore of invariable magnitude during the whole motion.

Poisson; *Traité de Mécanique*, Tom. II. p. 12.

(3) One body draws up another on the wheel and axle; to determine the motion of the weights and the tension of the strings.

Let a, a', denote the radii of the wheel and axle; m, m', the masses of the bodies suspended from them; s the arc described, at the end of the time t, by a molecule μ of the mass of the wheel and axle, r the distance of the molecule from the axis of rotation; x, x', the vertical distances, below the horizontal plane through the axis, of the masses m, m'.

Then the moment of the impressed forces about the axis of rotation will be

$$mag - m'a'g;$$

and the moments of the effective forces, estimated in the same direction, will be

$$ma\frac{d^2x}{dt^2} - m'a'\frac{d^2x'}{dt^2} + \Sigma\left(\mu r\frac{d^2s}{dt^2}\right).$$

Hence, by D'Alembert's Principle,

$$ma\frac{d^2x}{dt^2} - m'a'\frac{d^2x'}{dt^2} + \Sigma\left(\mu r\frac{d^2s}{dt^2}\right) = mag - m'a'g \text{............} (1).$$

Let θ represent the whole angle through which the wheel and axle have rotated at the end of the time t; then, b, b', denoting the initial values of x, x', it is clear that

$$x = b + a\theta, \quad x' = b' - a'\theta,$$

and therefore

$$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}, \qquad \frac{d^2x'}{dt^2} = -a'\frac{d^2\theta}{dt^2} \text{..................} (2).$$

Also, it is manifest that $s = r\theta$, and therefore

$$\Sigma\left(\mu r\frac{d^2s}{dt^2}\right) = \Sigma\left(\mu r^2\frac{d^2\theta}{dt^2}\right) = \frac{d^2\theta}{dt^2}\Sigma(\mu r^2) = Mk^2\frac{d^2\theta}{dt^2} \text{........} (3),$$

where Mk^2 denotes the moment of inertia of the wheel and axle together about the axis of rotation.

From (1), (2), (3), we obtain

$$(ma^2 + m'a'^2 + Mk^2)\frac{d^2\theta}{dt^2} = mag - m'a'g,$$

and therefore, if the system be supposed to have no motion when $t = 0$,

$$\theta = \tfrac{1}{2}gt^2\frac{ma - m'a'}{ma^2 + m'a'^2 + Mk^2} \text{....................} (4).$$

Let T denote the tension of the string supporting m; then

$$\begin{aligned}
T &= m\left(g - \frac{d^2x}{dt^2}\right)\\
&= m\left(g - a\frac{d^2\theta}{dt^2}\right)\\
&= mg\left\{1 - \frac{a(ma - m'a')}{ma^2 + m'a'^2 + Mk^2}\right\}\\
&= mg\frac{m'a'(a + a') + Mk^2}{ma^2 + m'a'^2 + Mk^2}.
\end{aligned}$$

Similarly, the tension of the other string being denoted by T',

$$T' = m'g\frac{ma(a' + a) + Mk^2}{m'a'^2 + ma^2 + Mk^2}.$$

(4) A thin uniform rod AB (fig. 180) slides down between the vertical and horizontal rods OBy, OAx, to which it is attached by small rings at A and B: to find the angular velocity of AB in any position.

Let X = the pressure of Oy on AB,

Y ~~Z~~ = ... Ox on AB.

Let m denote the mass of an elemental length ds of the rod at P: let $OM = x$, $PM = y$, $AB = a$, $\angle OAB = \theta$, $AP = s$.

The moving forces on m will be

the effective force $m\dfrac{d^2x}{dt^2}$, parallel to Ox,

the impressed force mg, parallel to yO,

the effective force $m\dfrac{d^2y}{dt^2}$, parallel to Oy.

Reverse the directions of $m\dfrac{d^2x}{dt^2}$ and $m\dfrac{d^2y}{dt^2}$, as in the figure: and let the same thing be done in regard to all the molecules of the descending rod. Then the system of forces will satisfy the conditions of equilibrium.

Hence
$$\Sigma\left(m\frac{d^2x}{dt^2}\right)=X \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\Sigma m\left(\frac{d^2y}{dt^2}+g\right)=Y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\Sigma\left\{m\left(\frac{d^2y}{dt^2}+g\right)x-m\frac{d^2x}{dt^2}y\right\}+Xa\sin\theta-Ya\cos\theta=0 \ldots (3).$$

Let λ denote the mass of a unit of length of the rod: then $m=\lambda ds$. Also

$$x=(a-s)\cos\theta, \qquad y=s\sin\theta.$$

Hence
$$\Sigma\left(m\frac{d^2x}{dt^2}\right)=\lambda\int_0^a(a-s)\,ds\frac{d^2\cos\theta}{dt^2}$$
$$=\tfrac{1}{2}\lambda a^2\frac{d^2\cos\theta}{dt^2},$$

$$\Sigma m\left(\frac{d^2y}{dt^2}+g\right)=\lambda\int_0^a\left(sds\frac{d^2\sin\theta}{dt^2}+gds\right)$$
$$=\lambda\left(\tfrac{1}{2}a^2\frac{d^2\sin\theta}{dt^2}+ga\right),$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$
$$=\lambda\frac{d^2\theta}{dt^2}\int_0^a s(a-s)\,ds$$
$$=\tfrac{1}{6}\lambda a^3\frac{d^2\theta}{dt^2},$$

$$\Sigma(mgx)=\lambda g\cos\theta\int_0^a(a-s)\,ds$$
$$=\tfrac{1}{2}\lambda ga^2\cos\theta.$$

The equations (1), (2), (3), therefore become, if M denote the mass of the whole rod,

$$X=\tfrac{1}{2}Ma.\frac{d^2\cos\theta}{dt^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$Y=M\left(g+\tfrac{1}{2}a\frac{d^2\sin\theta}{dt^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\tfrac{1}{6}Ma\frac{d^2\theta}{dt^2}+\tfrac{1}{2}Mg\cos\theta+X\sin\theta-Y\cos\theta=0 \ldots\ldots (6).$$

From (4), (5), (6), we shall get, after eliminating X and Y,

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2a}\cos\theta \dots\dots\dots\dots (7),$$

integrating, we have, α being the initial value of θ,

$$\frac{d\theta^2}{dt^2} = \frac{3g}{a}(\sin\alpha - \sin\theta) \dots\dots\dots\dots (8),$$

which determines the angular velocity of AB in any position.

COR. From (4), (5), (7), (8), we may easily ascertain that

$$X = \tfrac{3}{4} Mg \cos\theta\,(3\sin\theta - 2\sin\alpha),$$

$$Y = Mg - \tfrac{3}{4} Mg\,(1 + 2\sin\alpha\sin\theta + \sin^2\theta).$$

(5) A uniform heavy rod OA (fig. 181), which is at liberty to oscillate in a vertical plane about a horizontal axis through O, falls from a horizontal position; to determine the angle included between the direction of the rod and the direction of the pressure for any position of the rod.

Let Ox, Oy, be the axes of co-ordinates in the plane of oscillation, Ox being horizontal and Oy vertical; let Oz be at right angles to the plane xOy. Let U, V, represent the resolved parts of the reaction of the axis Oz upon the rod, estimated along xO, yO. Let ρ = the density of the rod, κ = the area of a section of it taken at right angles to its length; let P be any point in OA, draw PM at right angles to Ox; let

$$OM = x, \quad PM = y, \quad OP = r, \quad OA = a, \quad \angle AOx = \theta.$$

Then, by D'Alembert's Principle, resolving forces parallel to Ox,

$$U = -\int_0^a \left\{\kappa\rho\, dr \frac{d^2x}{dt^2}\right\} = -\kappa\rho\int_0^a \left(dr \frac{d^2x}{dt^2}\right) \dots\dots\dots\dots (1);$$

resolving forces parallel to Oy,

$$V = -\kappa\rho\int_0^a \left\{dr\left(\frac{d^2y}{dt^2} - g\right)\right\} \dots\dots\dots\dots (2);$$

and, taking moments about the axis Oz,

$$\int_0^a \kappa\rho\, g dr \,.\, x = \int_0^a \left\{\kappa\rho\, dr\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right\},$$

$$g\int_0^a x dr = \int_0^a \left\{dr\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right\} \dots\dots\dots\dots (3).$$

But, from the geometry, we see that

$$x = r\cos\theta,\ \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt},\ \frac{d^2x}{dt^2} = -r\cos\theta\frac{d\theta^2}{dt^2} - r\sin\theta\frac{d^2\theta}{dt^2},$$

and similarly

$$\frac{d^2y}{dt^2} = -r\sin\theta\frac{d\theta^2}{dt^2} + r\cos\theta\frac{d^2\theta}{dt^2};$$

hence, from (1), we have

$$U = \kappa\rho\int_0^a r dr\left(\cos\theta\frac{d\theta^2}{dt^2} + \sin\theta\frac{d^2\theta}{dt^2}\right)$$

$$= \tfrac{1}{2}a^2\kappa\rho\left(\cos\theta\frac{d\theta^2}{dt^2} + \sin\theta\frac{d^2\theta}{dt^2}\right)\text{..............}\ (4);$$

and, from (2),

$$V = \kappa\rho ga + \tfrac{1}{2}a^2\kappa\rho\left(\sin\theta\frac{d\theta^2}{dt^2} - \cos\theta\frac{d^2\theta}{dt^2}\right)\text{..............}(5).$$

Again, from (3), substituting for x and y their values in terms of r and θ, we get

$$g\int_0^a \cos\theta\, rdr = \int_0^a r^2 dr\frac{d^2\theta}{dt^2},$$

and therefore

$$\tfrac{1}{2}ga^2\cos\theta = \tfrac{1}{3}a^3\frac{d^2\theta}{dt^2},\qquad \frac{d^2\theta}{dt^2} = \frac{3g}{2a}\cos\theta;$$

multiplying by $2\frac{d\theta}{dt}$, integrating, and bearing in mind that $\theta = 0$ when $\frac{d\theta}{dt} = 0$, we have

$$\frac{d\theta^2}{dt^2} = \frac{3g}{a}\sin\theta.$$

Hence, substituting for $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ their values in (4) and (5), we obtain

$$U = \tfrac{9}{4}\kappa\rho\, ag\sin\theta\cos\theta,$$

$$V = \tfrac{1}{4}\kappa\rho\, ag\,(10 - 9\cos^2\theta).$$

From these equations we get

$$U\cos\theta + V\sin\theta = \tfrac{5}{2}\kappa\rho\, ag\sin\theta,$$

$$V\cos\theta - U\sin\theta = \tfrac{1}{4}\kappa\rho\, ag\cos\theta.$$

But $U\cos\theta + V\sin\theta$ and $V\cos\theta - U\sin\theta$ are the expressions for the resolved parts of the reaction of the fixed axis, estimated along AO and at right angles to AO; hence, if ϕ denote the inclination of the resultant reaction to the line AO produced, or of the resultant pressure on the axis to the line OA, we shall have

$$\tan\phi = \frac{V\cos\theta - U\sin\theta}{U\cos\theta + V\sin\theta} = \tfrac{1}{10}\cot\theta,$$

$$\tan\theta\tan\phi = \tfrac{1}{10}.$$

(6) A small body is suspended from a fixed point by a string, and is attracted towards a point, the distance of which from it is large compared with the length of the string: if the time of a small oscillation is proportional to the distance at which the attracting point is removed, to determine the law of the attracting force.

The attracting force varies inversely as the square of the distance.

(7) A flexible chain of uniform thickness moves upon two inclined planes, placed back to back; to find its tension at any point; also to find the greatest tension at the common summit of the planes, and to determine whether it is greater or less than the tension at the same point when there is equilibrium.

Let l denote the whole length of the chain, m the mass of a unit of its length; α, β, the inclinations of CP, CQ, the two portions of the chain, to the horizon; let $CP = r$; $T =$ the tension at any point E in CP; $CE = x$. Then

$$T = \frac{mg}{l}(r - x)(l - r)(\sin\alpha + \sin\beta):$$

the greatest tension at the common summit is equal to

$$\tfrac{1}{4}\,mgl(\sin\alpha + \sin\beta),$$

a less quantity than when there is equilibrium, unless $\alpha = \beta$.

(8) Two particles, connected together by a rigid imponderable rod, are constrained to move along two grooves Ox, Oy, respectively, the former horizontal, the latter vertical: supposing the

particles to be placed in any assigned position, to find the angular velocity of the rod in any position of its descent, and pressures on the grooves.

Let θ denote the inclination of the rod to the horizon at any time, ω the corresponding angular velocity, α the initial value of θ, l the length of the rod; X, Y, the pressures on the grooves Oy, Ox, respectively; m, m', the masses of the particles in the horizontal and vertical grooves respectively: then

$$\omega = \left(\frac{2m'g}{l}\right)^{\frac{1}{2}} \cdot \left(\frac{\sin\alpha - \sin\theta}{m\sin^2\theta + m'\cos^2\theta}\right)^{\frac{1}{2}},$$

$$X = \frac{mm'g\cos\theta}{(m\sin^2\theta + m'\cos^2\theta)^2} \cdot \{\sin\theta\,(m\sin^2\theta + m'\cos^2\theta) - 2m'(\sin\alpha - \sin\theta)\},$$

$$Y = mg + \frac{mm'g\sin\theta}{(m\sin^2\theta + m'\cos^2\theta)^2} \cdot \{\sin\theta\,(m\sin^2\theta + m'\cos^2\theta) - 2m'(\sin\alpha - \sin\theta)\}.$$

CHAPTER VII.

MOTION OF RIGID BODIES ABOUT FIXED AXES.

SECT. 1. *Various Problems.*

LET F denote the resolved part of any one of a system of forces acting on a rigid body, at right angles to a fixed axis, r being the perpendicular distance between the fixed axis and the direction of F. Then Fr will be the moment of this force about the axis, and, if $\Sigma\,(Fr)$ denote the sum of the moments of all the forces affected by their appropriate signs, we shall have, for the determination of the motion of the body, the general formula

$$\frac{d\omega}{dt} = \frac{\Sigma\,(Fr)}{Mk^2},$$

where $\omega =$ the angular velocity of the body after a time t, and $Mk^2 =$ its moment of inertia about the fixed axis.

(1) A straight uniform rod, moveable about its upper end, hangs vertically: to find the least angular velocity with which it must begin to move that it may perform complete revolutions in a vertical plane.

Let OA, (fig. 182), be the rod in any position; let $\theta =$ its inclination to the vertical line Ox at any time t. Let G be the centre of gravity: draw GH at right angles to Ox. Let $OA = a$, $m =$ the mass of the rod.

Then, for the motion,

$$mk^2 \frac{d^2\theta}{dt^2} = -\tfrac{1}{2} amg \sin\theta :$$

but $k^2 = \frac{1}{3}a^2$: hence

$$2a \frac{d^2\theta}{dt^2} = -3g \sin\theta,$$

$$a \frac{d\theta^2}{dt^2} = C + 3g \cos\theta :$$

let $\omega =$ the initial angular velocity of the rod: then

$$a\omega^2 = C + 3g,$$

and therefore

$$a\frac{d\theta^2}{dt^2} = a\omega^2 - 3g(1 - \cos\theta).$$

Again, the condition of the problem requires that $\frac{d\theta}{dt} = 0$ when $\theta = \pi$: hence

$$0 = a\omega^2 - 6g,$$

and therefore $$\omega = \left(\frac{6g}{a}\right)^{\frac{1}{2}}.$$

(2) A straight rod AB, (fig. 183), is freely moveable about its lower end A, which is fixed, while the other end B is suspended by a fine string BC attached to a fixed point C: when the system is slightly displaced from its position of equilibrium, so as to keep the string at full stretch, to find the time of a small oscillation.

Let $AB = a$; join CA; let $\alpha =$ the inclination of CA to the horizon, $\angle BAC = \epsilon$, $\theta =$ the inclination of the plane BAC to the vertical plane through AC, $mk^2 =$ the moment of inertia of AB about AC.

The component of the weight mg of the rod at right angles to AC is $mg \cos\alpha$, and the arm of the moment of this component about AC is $\frac{1}{2}a \sin\epsilon \,.\, \sin\theta$: hence, for the motion,

$$mk^2\frac{d^2\theta}{dt^2} = -mg\cos\alpha \,.\, \tfrac{1}{2}a\sin\theta \,.\, \sin\epsilon;$$

but $k^2 = \frac{1}{3}a^2\sin^2\epsilon$: hence, θ being small,

$$\frac{d^2\theta}{dt^2} + \frac{3g\cos\alpha}{2a\sin\epsilon}\,.\,\theta = 0.$$

Hence the time of an oscillation is equal to $\pi\left(\frac{2a\sin\epsilon}{3g\cos\alpha}\right)^{\frac{1}{2}}$.

(3) Supposing the force which acts on the crank of a steam-engine to be vertical, and to vary as the sine of the angle through which the crank has revolved at any time from a vertical position; to find the angular velocity of the crank in any position, the moment of the resistance being always equal to half

the greatest moment of the force, and the moment of the weight of the crank being regarded as inconsiderable.

Let AO (fig. 184) be the crank, O being the fixed extremity; draw Ox vertical; let $\angle AOx = \theta$ at any time t; $F =$ the force acting at the extremity A; $OA = a$: assume $F = \mu \sin\theta$; let mk^2 denote the moment of inertia of the crank about O.

Then, the moment of the resistance about O being $\frac{1}{2}\mu a$, we have, for the motion of the crank,

$$mk^2 \frac{d^2\theta}{dt^2} = \mu \sin\theta \, . \, a \sin\theta - \frac{1}{2}\mu a$$

$$= -\tfrac{1}{2}\mu a \cos 2\theta :$$

multiplying by $2\dfrac{d\theta}{dt}$ and integrating, we obtain

$$mk^2 \frac{d\theta^2}{dt^2} = C - \frac{1}{2}\mu a \sin 2\theta :$$

let ω denote the angular velocity of the crank when $\theta = 0$; then

$$mk^2\omega^2 = C;$$

hence
$$\frac{d\theta^2}{dt^2} = \omega^2 - \frac{\mu a \sin 2\theta}{2mk^2},$$

which gives the angular velocity of the crank in any position: from this result we see that the angular velocity is always ω when the crank is in either a horizontal or a vertical position.

SECT. 2. *Uniform Revolution.*

(1) An isosceles right-angled triangle ABC (fig. 185) is suspended at the right angle A, and its side AB is kept vertical by a ring at B; an angular velocity ω being communicated to the triangle round AB, to determine the magnitude of ω that there may be no pressure at B.

Bisect BC in L, join AL, and take $AG = \frac{2}{3}AL$; then G will be the centre of gravity of the triangle; draw GH at right angles to AC. Take P any point in the area of the triangle, and draw PM at right angles to AB. Let $AM = x$, $PM = y$, $AC = a = AB$; $m =$ the mass of a unit of area of the triangle.

Then

$$AH = AG\cos\frac{\pi}{4} = \frac{2}{3}AL\cos\frac{\pi}{4} = \frac{2}{3}a\left(\cos\frac{\pi}{4}\right)^2 = \frac{1}{3}a.$$

Also the area of the triangle is equal to $\frac{1}{2}a^2$, and therefore its mass to $\frac{1}{2}ma^2$; hence the moment of the triangle about an axis through A at right angles to its plane at any instant, in consequence of gravity, is

$$\tfrac{1}{2}ma^2g \cdot \tfrac{1}{3}a = \tfrac{1}{6}ma^3g.$$

Again, the moment about the same axis due to centrifugal force is equal to

$$\iint m\omega^2 y\,dx\,dy \,.\, x = \tfrac{1}{2}m\omega^2\int y^2 x\,dx$$

$$= \tfrac{1}{2}m\omega^2\int_0^a (a-x)^2\, x\,dx = \tfrac{1}{24}m\omega^2a^4.$$

Now, since there is no pressure on the ring at B, the moments of gravity and of centrifugal force about the axis through A must be equal; hence we have

$$\tfrac{1}{6}ma^3g = \tfrac{1}{24}m\omega^2a^4,$$

and therefore $$\omega^2 = \frac{4g}{a}, \quad \omega = 2\left(\frac{g}{a}\right)^{\frac{1}{2}}.$$

(2) A string lying in the form of a circle on a smooth table is revolving like a wheel: to find the tension of the string.

Let m = the mass of a unit of length of the string, mds = the mass of the element Pp (fig. 186): the moving force on the element due to rotation is equal to $mds \,.\, \omega^2 r$, ω being the angular velocity and r the radius.

Let t be the tension at P, the tension at p being accordingly $t + dt$. Resolving tangentially we have, $\angle POp$ being denoted by θ, for the equilibrium of Pp,

$$t = (t + dt)\cos\theta,$$

or, in the limit,

$$t = t + dt,$$

or $$dt = 0, \quad t = \text{a constant quantity}.$$

To find this constant value we have, resolving normally,

$$m\,ds\,.\,\omega^2 r = (t + dt)\sin\theta = t\theta, \text{ in the limit:}$$

whence,

$$t = mr^2\omega^2,$$

or the tension varies as the square of the angular velocity.

(3) Two equal uniform rods AB, AC (fig. 187) are connected at one extremity A by a hinge, the other extremities being connected by a fine string BC: they are whirled round with a given angular velocity, so that the axis of the isosceles triangle formed by the string and rods is always vertical; to find the tension of the string.

Let $AB = 2a$, $T =$ the tension of the string, $W =$ the weight of either rod, $\omega =$ the angular velocity about the vertical axis AE of the triangle, $\angle BAE = \alpha$. Take P any point in AB; let $AP = r$.

Then, taking moments of the forces acting upon AB, about the point A, we have

$$T\,.\,2a\cos\alpha = Wa\sin\alpha + \int_0^{2a} \omega^2 r \sin\alpha\,.\,\frac{W\,dr}{2ag}\,.\,r\cos\alpha$$

$$= Wa\sin\alpha + \frac{W\omega^2 \sin\alpha\cos\alpha}{2ag}\,.\,\frac{8}{3}a^3$$

$$= Wa\sin\alpha\left(1 + \frac{4a\omega^2\cos\alpha}{3g}\right),$$

$$T = \tfrac{1}{2} W \tan\alpha\,.\left(1 + \frac{4a\omega^2\cos\alpha}{3g}\right).$$

(4) A rod AB, the length of which is $2a$ and weight W, has one extremity attached to a hinge A in a vertical axis, and, at the other extremity B, is connected with a weight P by means of a fine string passing through a small hole in the axis at the distance $2a$ above A; supposing the axis and rod to revolve so as to form a constant angle $\frac{\pi}{4}$, to determine the angular velocity.

If $\omega =$ the angular velocity,

$$\omega^2 = \frac{3g}{2a\sqrt{2}}\left\{1 - \frac{P}{W}\sqrt{(4 - \sqrt{8})}\right\}.$$

(5) A carriage moves on a railroad with a given velocity round a curve of given radius: to find the amount by which the outer rail must be elevated above the inner one in order that the carriage should not be overturned towards the outside.

We will suppose the radii of the circles described by the molecules of the carriage to be the same, as will be approximately the case in railroads.

Let $2b$ = the breadth of the road between the rails, a = the distance of the centre of gravity of the carriage from the road, r = the radius of the curve, v = the velocity of each molecule of the carriage, and θ = the inclination of the road to the horizon: then

$$\tan\theta = \frac{av^2 - bgr}{bv^2 + agr}.$$

(6) A thin book lies on one of the faces of a desk; to find the greatest angular velocity round a vertical axis which can be given to the desk without throwing off the book.

Let α = the inclination of the desk to the horizon, a = the length of the book; and, the book being supposed to be placed symmetrically on one face of the desk, let c = the distance of its lower edge from the axis of revolution, ω = the required angular velocity. Then, the book being supposed to be moveable about its lower edge, which is kept at rest by the ledge of the desk,

$$\omega^2 = \frac{3g\cot\alpha}{3c - 2a\cos\alpha}.$$

(7) A Ring, surrounding a Planet, revolves uniformly about a diameter passing through the common centre of the Ring and the Planet: to determine the form of the Ring in order that the tangential stress may be the same at all points.

If ω = the angular velocity, μ = the attraction of the planet at a unit of distance, $2a$ = the diameter of revolution; then, the prime radius vector being supposed to be coincident with the diameter of revolution, the equation to the ring will be

$$\frac{1}{a} - \frac{1}{r} = \frac{\omega^2}{2\mu} r^2 \sin^2\theta.$$

SECT. 3. *Centre of Oscillation.*

Conceive a body of any figure, acted on by gravity, to be oscillating about a fixed horizontal axis AB (fig. 188); let G be the centre of gravity of the body; draw GO at right angles to AB. Produce OG to a point C such that

$$OC = \frac{h^2 + k^2}{h},$$

where $h = OG$ and $k =$ the radius of gyration of the body about an axis through G parallel to AB; then, if the whole mass of the body be collected at the point C, the period of its oscillations about AB will be the same as before. The point C is called the Centre of Oscillation or of Agitation.

The theory of the Centre of Oscillation of bodies originated in questions addressed, about the year 1646, by Mersenne to the mathematicians of his day, who were called upon by him to exert their ingenuity to discover the time of oscillation of bodies moveable about horizontal axes. It is rather singular that all those who first attempted the solution of this celebrated problem, among whom Mersenne[1] himself is to be numbered, together with Descartes[2], Roberval[3], Wallis[4], and Fabri[5], tacitly supposed the Centre of Oscillation to be coincident with the Centre of Percussion; a supposition which, although true, is by no means obvious without a rigorous demonstration. On the strength of this assumption, however, the Centre of Oscillation was correctly determined in the case of certain figures. Descartes gave a true solution of the case where a plane area oscillates *in planum*, but failed in the case of solid bodies and of plane areas oscillating *in latus*. Roberval assigned correctly the position of the Centre of Oscillation, not only of plane areas oscillating *in planum*, but also in certain instances of oscillation *in latus*, while together with Descartes he failed to give a correct

[1] *Mersenni Reflexiones Physico-Mathematicæ*, Cap. XI. et XII.
[2] *Lettres de Descartes*, Tom. III. p. 487, &c.
[3] *Lettres de Descartes*, ib.
[4] *Mechanica, sive De Motu.*
[5] *Tract. de Motu, Append. Physico-Math. De Centro Percussionis.*

solution of the problem in the case of solid figures. The labours of Huyghens, who in his earlier efforts to obtain a solution of Mersenne's problem had been utterly baffled, were at length crowned with success, and accordingly in the fourth part of his *Horologium Oscillatorium*, which appeared in the year 1673, was given the first rigorous and general investigation of the Centre of Oscillation. The two following axioms constitute the basis of his researches: first, that the centre of gravity of a system of heavy bodies cannot of itself rise to an altitude greater than that from which it has fallen, whatever change be made in the mutual disposition of the bodies; and secondly, that a compound pendulum will always ascend to the same height as that from which it has descended freely. Some years after the publication of the *Horologium Oscillatorium*, the truth of these fundamental axioms, which although true, it must be admitted, are not sufficiently elementary, was called in question by the Abbé Catelan[1], who substituted certain frail theories of his own in place of the valuable researches of Huyghens. The attention of the mathematicians of the day having been more closely directed to the subject by the controversy which arose between Huyghens and Catelan, the views of Huyghens received ample corroboration from the more elementary investigations of L'Hôpital, James Bernoulli, and other mathematicians. For information respecting the subsequent history of Mersenne's problem, the reader is referred to the Chapter on D'Alembert's Principle.

(1) To find at what point of the rod of a perfect pendulum must be fixed a given weight of indefinitely small volume, so as to have the greatest effect in accelerating the pendulum.

Let m be the mass of the bob of the perfect pendulum, and a its length; m' the mass of the given weight, and a' the distance of its point of attachment from the centre of suspension; l the distance between the centre of suspension and the centre of oscillation of the complex pendulum. Then we shall have, m and m' being both of indefinitely small volume,

$$l = \frac{ma^2 + m'a'^2}{ma + m'a'}.$$

[1] *Journal des Sçavans*, 1682 *et* 1684.

Now the shorter the rod of a perfect pendulum, the shorter will be the time of its oscillations: hence we must have l a minimum; differentiating then with respect to a' we get

$$\frac{dl}{da'}=\frac{2m'a'(ma+m'a')-m'(ma^2+m'a'^2)}{(ma+m'a')^2}=0;$$

hence

$$m'a'^2+2maa'=ma^2,$$

$$m'^2a'^2+2ma\,m'a'+m^2a^2=(m^2+mm')\,a^2,$$

$$m'a'+ma=(m^2+mm')^{\frac{1}{2}}a,$$

$$a'=\frac{a}{m'}\left\{(m^2+mm')^{\frac{1}{2}}-m\right\},$$

which determines the required point of attachment.

> *Lady's and Gentleman's Diary*, 1742. *Diarian Repository*, p. 394. Euler; *De Motu Corp. Solid.*, Prob. 48. Cor. 1. p. 216.

(2) To compare the times in which a circular plate will vibrate round a horizontal tangent and round a horizontal axis, through the point of contact, at right angles to the tangent.

Let l, l', denote the lengths of the isochronous pendulums in the former and latter case respectively; a the radius of the plate; k, k', the radii of gyration about axes through the centre of the plate parallel in each case to the axis of oscillation. Then

$$l=\frac{a^2+k^2}{a}, \qquad l'=\frac{a^2+k'^2}{a}.$$

Let A denote the area of the plate; r the distance of a point within it from its centre, and θ the inclination of this distance to the horizon when the plate is hanging at rest. Then

$$Ak^2=\iint rd\theta dr\,.\,r^2\sin^2\theta,\ \text{between the proper limits,}$$

$$=\int_0^{2\pi}\int_0^a r^3\sin^2\theta\,d\theta\,dr=\tfrac{1}{4}a^4\int_0^{2\pi}\sin^2\theta\,d\theta=\tfrac{1}{4}\pi a^4.$$

Also

$$Ak'^2=\iint rd\theta dr\,.\,r^2=\int_0^{2\pi}\int_0^a r^3d\theta dr=\tfrac{1}{4}a^4\int_0^{2\pi}d\theta=\tfrac{1}{2}\pi a^4.$$

But $A = \pi a^2$; hence $k^2 = \frac{1}{4} a^2$ and $k'^2 = \frac{1}{2} a^2$; and therefore

$$l = a + \tfrac{1}{4} a = \tfrac{5}{4} a, \qquad l' = a + \tfrac{1}{2} a = \tfrac{6}{4} a.$$

Hence, if t, t', denote the times of vibration,

$$\frac{t}{t'} = \left(\frac{l}{l'}\right)^{\frac{1}{2}} = \left(\frac{5}{6}\right)^{\frac{1}{2}}.$$

(3) To find the length of a simple pendulum oscillating in the same time as the arc of a given circle, the axis of oscillation passing through the middle point of the arc at right angles to its plane.

Let C, (fig. 189), be the centre of the circle, A the middle point of the arc, P any point in the arc. Draw PM at right angles to AC: let $AM = x$, $AC = a$, $\delta m =$ the mass of an element of the arc at P, $l =$ the length of the required pendulum. Then

$$l = \frac{\Sigma\,(r^2 \delta m)}{\Sigma\,(x\delta m)} = \frac{\Sigma\,(2ax\delta m)}{\Sigma\,(x\delta m)} = 2a,$$

a result which shews that the length of the simple pendulum depends only upon the radius of the circle, and not upon the length of the arc. *Lady's Diary*, 1841.

(4) If l and h be the distances of the centres of oscillation and gravity of a mercurial pendulum of which the weight is m, from the axis of suspension, and h' be the distance of the centre of gravity of a small quantity of mercury μ by the addition of which the pendulum is made to vibrate seconds exactly, to determine the approximate ratio of μ to m, L being the length of the seconds pendulum, and r the radius the cylinder containing the mercury.

The moment of inertia of the mercury μ, which may be regarded approximately as a circular lamina of fluid, about any diameter, and therefore about a diameter parallel to the axis from which the pendulum is suspended, will be $\frac{1}{4}\mu r^2$, and therefore its moment of inertia about the axis of suspension will be

$$\mu\,(h'^2 + \tfrac{1}{4} r^2).$$

Also, the radius of gyration of the mercury m about a line through its centre of gravity parallel to the axis of suspension being k, the moment of inertia about the axis of suspension will be $m(h^2+k^2)$. Hence, by the formula for the Centre of Oscillation, we have approximately

$$(\mu h' + mh) L = \mu (h'^2 + \tfrac{1}{4} r^2) + m (a^2 + k^2).$$

But also we shall have

$$hl = h^2 + k^2;$$

hence
$$(\mu h' + mh) L = \mu (h'^2 + \tfrac{1}{4} r^2) + mhl,$$

$$\frac{\mu}{m} \{h' (L - h') - \tfrac{1}{4} r^2\} = h (l - L),$$

$$\frac{\mu}{m} = \frac{4h (l - L)}{4h' (L - h') - r^2}.$$

(5) A bent lever, of which the arms are of lengths a and b, and the angle between them θ, makes small oscillations in its own plane about the angular point; to find the length of the isochronous simple pendulum.

$$\text{The required length} = \frac{\tfrac{2}{3} (a^3 + b^3)}{(a^4 + 2a^2b^2 \cos\theta + b^4)^{\frac{1}{2}}}.$$

(6) A bent lever, the arms of which are of equal weight, and which are inclined to each other at right angles, makes small oscillations in its own plane about its angular point: to find the length of the isochronous simple pendulum.

The length of the required pendulum is equal to four-thirds of the diameter of a circle of which the arms of the lever are chords.

(7) To ascertain at what point in its length a uniform straight rod of small thickness must be suspended that it may oscillate isochronously with a given simple pendulum a.

Let $2a =$ the length of the rod, $l =$ the length of the given pendulum, $h =$ the distance of the required point of suspension from the rod's centre of gravity. Then

$$h = \tfrac{1}{2}\,l \pm (\tfrac{1}{4}\,l^2 - \tfrac{1}{3}\,a^2)^{\frac{1}{2}},$$

which shews that $\dfrac{2a}{\sqrt{3}}$ is the least admissible value of l.

(8) A heavy circular arc, of which the radius is a, and which subtends an angle 2α at the centre of the circle, oscillates, in a vertical plane, between two inclined planes: to find the length of the isochronous simple pendulum.

The required length is equal to $\dfrac{a\alpha}{\sin\alpha}$.

(9) To investigate the form of an isosceles triangle, the oscillations of which may have the same amplitude and period round an axis, perpendicular to its plane, through its vertex, and round an axis, parallel to the former, through the middle point of its base.

The vertical angle of the triangle must be a right angle.

(10) A square oscillates about a horizontal axis perpendicular to its plane: to find where the axis must pierce the square that the time of oscillation may be a minimum.

If c = the length of a side of the square, the locus of the required point is a circle described about the centre of the square with a radius $\dfrac{c}{\sqrt{6}}$.

(11) A square lamina oscillates flat-ways about a horizontal axis passing through one of its angular points: to find the length of the isochronous simple pendulum.

The required length $= \frac{7}{12} \times$ diagonal.

(12) A sector of a circle oscillates round a horizontal axis at right angles to its plane through the centre of the circle; to find the angle of the sector when the length of the isochronous simple pendulum is equal to one half the length of the arc.

If ϕ = the angle of the sector,

$$\cos\phi = -\tfrac{1}{8}.$$

(13) A uniform rod of given length is bent into the form of a cycloid, and oscillates about a horizontal line joining its extremities; to find the length of the isochronous pendulum.

If a be the length of the rod, the length of the isochronous pendulum will be $\frac{1}{5} a$.

(14) A pendulum consists of an indefinitely thin rigid rod OA, and a globe of which the centre is A; to determine the point A', in the line OA, at which the centre of another globe must be fixed in order that the oscillations of the system of the two globes may be executed in the smallest time possible.

Let $OA = a$, $OA' = a'$; also let r, r', be the radii, and m, m', the masses of the globes A, A'. Then

$$a' = \frac{1}{m'} \left\{ m (m + m') a^2 + \tfrac{2}{5} m' (mr^2 + m'r'^2) \right\}^{\frac{1}{2}} - \frac{ma}{m'}.$$

Euler; *Theoria Motus Corporum Solidorum*, p. 215.

CHAPTER VIII.

MOTION OF RIGID BODIES. FREE AXES. SMOOTH SURFACES.

If a body be in motion about a Principal Axis[1], and be acted on by forces which do not tend to perturb the direction of this axis; then, the motion of the centre of gravity of the body remaining the same as if all the forces were impressed on the mass condensed at this point, the Principal Axis will always remain parallel to itself as an axis of permanent rotation, and the angular acceleration about this axis will be the same as if it were a fixed axis. The discovery of the existence of three principal axes in every body as axes of permanent rotation is due to Professor Segner of Gottingen, by whom it was communicated to the world in a memoir entitled *Specimen Theoriæ Turbinum*, published at Halle in the year 1755. For the complete development of the theory of rotation about permanent axes, the student is referred to Euler's *Theoria Motus Corporum Solidorum*, cap. VIII., a work of the greatest value for those who wish to acquire profound views on the subject of the motion of rigid bodies.

If a body be revolving at any instant of time about an axis which is not a principal one, this axis will not be one of permanent rotation; the body will revolve successively about a series of instantaneous axes, the positions of which both in relation to the body and to absolute space are different. The solution of the great physical problem of the Precession of the Equinoxes, published by D'Alembert[2] in the year 1749, unfolded a complete method for the investigation of the general problem of

[1] The following is the definition of Principal Axes given by Euler, *Theoria Motus Corporum Solidorum*, p. 175: "Axes principales cujusque corporis sunt tres illi axes per ejus centrum inertiæ transeuntes, quorum respectu momenta inertiæ sunt vel maxima vel minima."

[2] *Recherches sur la Précession des Equinoxes*, 1749.

rotation. In the following year was published by Euler[1] a memoir entitled *Découverte d'un nouveau principe de Méchanique*, the object of which was to investigate general formulæ for the motion of a body under the most general circumstances of motion and force. The equations, however, expressing under the most simple form the general conditions of rotation, were first given by Euler[2] in the year 1758, who availed himself of the principles of simplification afforded by the recent discoveries of Segner[3] respecting the existence of the three Principal Axes of material bodies. The consideration of the general problem of rotation was resumed by D'Alembert, and presented under its most general aspect in the first volume of his *Opuscules Mathematiques*, published in 1761, where he expresses disapprobation of the title prefixed by Euler to his memoir of 1749, in consideration of his own investigations on the Precession of the Equinoxes. The subject of rotation was thoroughly investigated and exemplified by Euler in his *Theoria Motus Corporum Solidorum et Rigidorum*, which appeared in the year 1767. The same subject was afterwards investigated by Lagrange[4] on more general principles of analysis. In the year 1777 appeared a memoir entitled '*A new Theory of the Rotatory Motion of Bodies affected by Forces disturbing such motion*,' by Landen[5], a celebrated English mathematician, in which he expresses himself dissatisfied with the conclusions of the great continental philosophers on the subject of rotation. The subject was again resumed by Landen[6] a few years afterwards, when he developes more fully his own views, and persists in his opposition to the doctrines of his predecessors. There is a memoir by Wildbore in the *Philosophical Transactions* for the year 1790, in which the subject is investigated under a new light: the conclusions of the author are unfavourable to the cause of Landen, whose views are in fact now generally exploded. For further information on the history of the theory

[1] *Mémoires de Académie des Sciences de Berlin*, 1750.

[2] *Ibid.* 1758.

[3] *Specimen Theoriæ Turbinum*, 1755.

[4] *Mémoires de l'Académie des Sciences de Berlin*, 1753; *Mécanique Analytique*, Seconde Partie, Section IX.

[5] *Philosophical Transactions*, 1777.

[6] *Ibid.* 1785.

of rotation and Landen's controversy, the student is referred to a memoir by Mr. Whewell, in the second volume of the *Cambridge Philosophical Transactions*, 1827. The investigation of Euler's general equations of rotatory motion has been effected with great elegance and simplicity by Mr. O'Brien, in the fifth chapter of his *Mathematical Tracts*, part I.

SECT. 1. *Single Body.*

(1) A rod PQ (fig. 190) of uniform thickness and density, having been placed in a given position with one end upon a smooth horizontal plane OA, and the other leaning against a smooth vertical plane OB, descends in a vertical plane AOB by the action of gravity; to determine where the rod will detach itself from the vertical plane.

Let $PG = a = GQ$, G being the centre of gravity of the rod: let GH be vertical and equal to y at any time t of the motion; $OH = x$, $\angle QPO = \phi$; $k =$ the radius of gyration of the rod about G; $R =$ the reaction of the vertical plane, which will be horizontal, and $S =$ that of the horizontal plane, which will be vertical; $m =$ the mass of the rod.

Then, for the motion of the rod, we have, resolving forces horizontally,

$$m\frac{d^2x}{dt^2} = R \text{..........................} (1);$$

resolving vertically,

$$m\frac{d^2y}{dt^2} = S - mg \text{...................} (2);$$

and, taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2} = Ra\sin\phi - Sa\cos\phi \text{.................} (3).$$

Eliminating R and S between the three equations (1), (2), (3), we have

$$k^2\frac{d^2\phi}{dt^2} = a\sin\phi\frac{d^2x}{dt^2} - a\cos\phi\frac{d^2y}{dt^2} - ag\cos\phi \text{.....} (4).$$

Now, from the geometry, it is clear that

$$x = a\cos\phi, \quad y = a\sin\phi,$$

and therefore

$$\frac{dx}{dt} = -a \sin\phi \frac{d\phi}{dt}, \quad \frac{d^2x}{dt^2} = -a \cos\phi \frac{d\phi^2}{dt^2} - a \sin\phi \frac{d^2\phi}{dt^2}, \ldots (5),$$

and

$$\frac{dy}{dt} = a \cos\phi \frac{d\phi}{dt}, \quad \frac{d^2y}{dt^2} = -a \sin\phi \frac{d\phi^2}{dt^2} + a \cos\phi \frac{d^2\phi}{dt^2};$$

hence we have

$$a \sin\phi \frac{d^2x}{dt^2} - a \cos\phi \frac{d^2y}{dt^2} = -a^2 \frac{d^2\phi}{dt^2};$$

and therefore, from (4),

$$(a^2 + k^2) \frac{d^2\phi}{dt^2} = -ag \cos\phi \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

multiplying by $2\frac{d\phi}{dt}$, and integrating, we have

$$(a^2 + k^2) \frac{d\phi^2}{dt^2} = C - 2ag \sin\phi;$$

but, if α be the initial value of ϕ, we have, since $\frac{d\phi}{dt} = 0$ initially,

$$0 = C - 2ag \sin\alpha;$$

and therefore $\quad (a^2 + k^2) \frac{d\phi^2}{dt^2} = 2ag (\sin\alpha - \sin\phi) \ldots\ldots\ldots (7).$

Now, at the instant when the rod detaches itself from the vertical plane, $R = 0$; hence, by (1) and the value of $\frac{d^2x}{dt^2}$ in (5),

$$\cos\phi \frac{d\phi^2}{dt^2} + \sin\phi \frac{d^2\phi}{dt^2} = 0;$$

and therefore, by (6) and (7),

$$2ag (\sin\alpha - \sin\phi) \cos\phi = ag \cos\phi \sin\phi;$$

whence, since ϕ cannot be equal to $\frac{1}{2}\pi$, we have

$$2 \sin\alpha - 2 \sin\phi = \sin\phi, \quad \sin\phi = \tfrac{2}{3} \sin\alpha;$$

which gives the position of the rod at the moment of its separation from the vertical plane.

This problem was proposed by Weston, a disciple of Landen's, in the *Lady's and Gentleman's Diary* for the year 1757; and

solved by Peter Walton, a contributor to the *Diary*. See *Diarian Repository*, p. 467.

(2) A uniform rod of given length hangs horizontally by two equal vertical strings attached to its ends; if it be twisted horizontally through a very small angle, so that its centre of gravity remains in the same vertical line, to find the time of an oscillation, the inertia of the strings being neglected.

Let P, Q, (fig. 191), be the points from which the strings PA, QB, are suspended, AB being the position in which the rod will rest; let ab be the position of the rod at any instant after disturbance; G the centre of gravity of AB, and therefore approximately of ab. Let $AG = a = BG$, $AP = b = BQ$, $\angle AGa = \theta$, $m =$ the mass of the rod.

Then, for small oscillations, the tension of each string may be considered equal to $\frac{1}{2}mg$ and Aa equal to $a\theta$. Also the resolved part of the tension of aP along aA will be nearly equal to

$$\tfrac{1}{2}mg\frac{a\theta}{b} = \frac{mag\theta}{2b},$$

and its moment about G will be nearly equal to

$$\frac{ma^2g\theta}{2b};$$

similarly for the tension of the string bQ: hence for the angular motion of ab about G, taking into account the tensions of both the strings,

$$mk^2\frac{d^2\theta}{dt^2} = -\frac{ma^2g\theta}{b};$$

but $k^2 = \frac{1}{3}a^2$; hence

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{b}\theta;$$

hence the time of an oscillation will be equal to $\left(\dfrac{b}{3g}\right)^{\frac{1}{2}}\pi$;

which is that of a simple pendulum, of which the length is $\frac{1}{3}b$, and is independent of the length of the rod.

Lady's and Gentleman's Diary, 1842, p. 51.

(3) A heterogeneous sphere is placed upon a perfectly smooth horizontal plane, its centre of gravity being slightly distant from the vertical through its geometrical centre; to find the time of the small oscillation of the centre of gravity about the geometrical centre.

Let ABS (fig. 192) be a vertical section of the sphere passing through C its geometrical centre and G its centre of gravity. Draw GQ horizontal, intersecting the vertical line SCK, through the point of contact of the sphere and the horizontal plane, in the point Q; draw GM vertical to cut the horizontal plane in M; and let CGA be the radius through G. Let $\angle AGM = \phi = \angle ACS$; $CG = c$, $MG = y$, $m =$ the mass of the sphere, $k =$ the radius of gyration about G, $R =$ the reaction of the plane at S upon the sphere, which will exert itself vertically.

Then for the motion of the sphere we have, resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2} = -Rc\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots(2);$$

but $y = a - c\cos\phi$, and therefore, from (1),

$$R = mg - mc\frac{d^2\cos\phi}{dt^2};$$

hence from (2) we have

$$mk^2\frac{d^2\phi}{dt^2} = -mcg\sin\phi + mc^2\sin\phi\frac{d^2\cos\phi}{dt^2},$$

and therefore

$$(k^2 + c^2\sin^2\phi)\frac{d^2\phi}{dt^2} + c^2\sin\phi\cos\phi\frac{d\phi^2}{dt^2} = -cg\sin\phi;$$

multiplying this equation by $2\dfrac{d\phi}{dt}$, and integrating, we get

$$(k^2 + c^2\sin^2\phi)\frac{d\phi^2}{dt^2} = 2cg\cos\phi + C:$$

suppose α to be the initial value of ϕ; then, $\dfrac{d\phi}{dt}$ being supposed to be initially zero, we have

$$0 = 2cg\cos\alpha + C,$$

and therefore

$$(k^2 + c^2\sin^2\phi)\,\frac{d\phi^2}{dt^2} = 2cg\,(\cos\phi - \cos\alpha), \,\ldots\ldots\ldots(3),$$

whence, $\frac{dt}{d\phi}$ being considered negative because as t increases $d\phi$ is negative from the beginning to the end of every complete oscillation,

$$\frac{dt}{d\phi} = -\frac{1}{(2cg)^{\frac{1}{2}}}\,\frac{(k^2 + c^2\sin^2\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}};$$

now from (3) we see that when $\frac{d\phi}{dt} = 0$, $\cos\phi = \cos\alpha$, and therefore $\phi = \pm\alpha$, the positive value of ϕ corresponding to the beginning and the negative value to the end of a complete oscillation; hence, if T denote the time of a complete oscillation,

$$T = -\frac{1}{(2cg)^{\frac{1}{2}}}\int_{+\alpha}^{-\alpha} d\phi\,\frac{(k^2 + c^2\sin^2\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}$$

$$= \frac{1}{(2cg)^{\frac{1}{2}}}\int_{-\alpha}^{+\alpha} d\phi\,\frac{(k^2 + c^2\sin^2\phi)^{\frac{1}{2}}}{(\cos\phi - \cos\alpha)^{\frac{1}{2}}}.$$

Assume

$$\sin\frac{\phi}{2} = s, \quad \sin\frac{\alpha}{2} = b;$$

then

$$\tfrac{1}{2}\cos\frac{\phi}{2}\,d\phi = ds, \quad d\phi = \frac{2ds}{(1-s^2)^{\frac{1}{2}}},$$

$$\cos\phi - \cos\alpha = 2\,(b^2 - s^2), \quad \sin^2\phi = 4s^2\,(1 - s^2);$$

hence we have

$$T = \frac{1}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\,\frac{(1 - s^2)^{-\frac{1}{2}}\,\{k^2 + 4c^2s^2\,(1 - s^2)\}^{\frac{1}{2}}}{(b^2 - s^2)^{\frac{1}{2}}},$$

or, neglecting as inconsiderable powers of the small quantity s beyond the second,

$$= \frac{k}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\,\frac{1 + \frac{1}{2}s^2 + \frac{2c^2}{k^2}s^2}{(b^2 - s^2)^{\frac{1}{2}}}$$

$$=\frac{k}{(cg)^{\frac{1}{2}}}\int_{-b}^{+b} ds\left\{\frac{1}{(b^2-s^2)^{\frac{1}{2}}}+\frac{4c^2+k^2}{2k^2}\frac{s^2}{(b^2-s^2)^{\frac{1}{2}}}\right\};$$

but
$$\int_{-b}^{+b}\frac{ds}{(b^2-s^2)^{\frac{1}{2}}}=\pi,\qquad \int_{-b}^{+b}\frac{s^2ds}{(b^2-s^2)^{\frac{1}{2}}}=\tfrac{1}{2}\pi b^2;$$

hence we have, for the time of a complete oscillation,

$$T=\frac{\pi k}{(cg)^{\frac{1}{2}}}\left(1+\frac{4c^2+k^2}{4k^2}b^2\right)=\frac{\pi k}{(cg)^{\frac{1}{2}}}\left(1+\frac{4c^2+k^2}{4k^2}\sin^2\frac{\alpha}{2}\right).$$

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(4) A beam AB (fig. 193) is placed with one end A upon a smooth inclined plane EF: to find the motion of the beam and its pressure on the plane at any time.

Let G be the position of the centre of gravity of the beam at any time t from the commencement of the motion, $R=$ the reaction of the plane upon the extremity A, $\angle BAF=\phi$; let E be the initial position of A, and β the initial value of ϕ; $m=$ the mass of the beam, $k=$ its radius of gyration about G, $EA=z$, $EH=x$, $GH=y$, $\alpha=$ the inclination of FE to the horizon.

Then for the motion of the beam we have, resolving forces parallel to the plane,

$$m\frac{d^2x}{dt^2}=mg\sin\alpha\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

resolving forces at right angles to the plane,

$$m\frac{d^2y}{dt^2}=R-mg\cos\alpha\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, taking moments about G,

$$mk^2\frac{d^2\phi}{dt^2}=-Ra\cos\phi\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1) we get

$$\frac{dx}{dt}=gt\sin\alpha+C;$$

but $\frac{dx}{dt}=0$ when $t=0$; and therefore $C=0$; hence

$$\frac{dx}{dt}=gt\sin\alpha;$$

integrating, and observing that $x=a\cos\beta$ when $t=0$, we have

$$x=\tfrac{1}{2}gt^2\sin\alpha+a\cos\beta \qquad (4),$$

which gives the position of the point H at any assigned time from the commencement of the motion.

Again, from (2) and (3), by the elimination of R,

$$a\cos\phi\frac{d^2y}{dt^2}=-k^2\frac{d^2\phi}{dt^2}-ag\cos\alpha\cos\phi;$$

but, by the geometry, we see that $y=a\sin\phi$; hence

$$a^2\cos\phi\frac{d^2\sin\phi}{dt^2}=-k^2\frac{d^2\phi}{dt^2}-ag\cos\alpha\cos\phi;$$

and therefore, multiplying by $2\frac{d\phi}{dt}$, and integrating,

$$a^2\left(\frac{d\sin\phi}{dt}\right)^2=C-k^2\frac{d\phi^2}{dt^2}-2ag\cos\alpha\sin\phi;$$

but, initially, $\frac{d\phi}{dt}=0$ and $\phi=\beta$; hence there is

$$0=C-2ag\cos\alpha\sin\beta,$$

and therefore

$$(a^2\cos^2\phi+k^2)\frac{d\phi^2}{dt^2}=2ag\cos\alpha\,(\sin\beta-\sin\phi) \qquad (5),$$

which gives the angular velocity of the beam for every position which it can assume during its descent.

From the geometry it is evident that

$$z=x-a\cos\phi$$
$$=\tfrac{1}{2}gt^2\sin\alpha+a(\cos\beta-\cos\phi), \text{ by } (4),$$

and $\quad y=a\sin\phi;$

if therefore from (5) we could obtain ϕ in terms of t, we might determine the values of y and z at any time from the beginning of the motion.

Again, for the pressure on the plane at any time, we have, from (3),

$$R = -\frac{mk^2}{a\cos\phi}\frac{d^2\phi}{dt^2};$$

but, from (5),

$$\frac{d\phi^2}{dt^2} = 2ag\cos\alpha\,\frac{\sin\beta - \sin\phi}{a^2\cos^2\phi + k^2},$$

and therefore, differentiating with respect to t, and dividing by $2\frac{d\phi}{dt}\cos\phi$,

$$\frac{1}{\cos\phi}\frac{d^2\phi}{dt^2} = \frac{2a^3g\cos\alpha\sin\phi\,(\sin\beta - \sin\phi)}{(a^2\cos^2\phi + k^2)^2} - \frac{ga\cos\alpha}{a^2\cos^2\phi + k^2};$$

hence $R = \dfrac{mk^2g\cos\alpha}{a^2\cos^2\phi + k^2} - \dfrac{2ma^2k^2g\cos\alpha\sin\phi\,(\sin\beta - \sin\phi)}{(a^2\cos^2\phi + k^2)^2}$

$$= \frac{mk^2g\cos\alpha}{(a^2\cos^2\phi + k^2)^2}\{k^2 + a^2(1 + \sin^2\phi - 2\sin\beta\sin\phi)\},$$

which gives the pressure on the plane for any of the successive positions of the beam.

Fuss; *Nova Acta Petrop.* 1795; p. 70.

(5) A cylinder KLM, (fig. 194), is placed with its axis horizontal upon a smooth inclined plane; a string $EPMKL$, one end E of which is attached to a fixed point at a distance EA from the plane equal to the radius of the cylinder, having been wound about the cylinder in a vertical plane through the centre of gravity O of the cylinder at right angles to its axis; to find the tension of the string and the velocity of decrease of its angle of inclination to the plane corresponding to any position of the cylinder in its descent; the length of the free string being initially equal to zero.

Let M be the point of contact of the section KLM of the cylinder, about which the string is wound, with the inclined plane; and P the point in which the free string EP touches the cylinder. Produce EP to meet the inclined plane in S; join OP, OM; at any time t from the commencement of the motion let $AM = x$, $T =$ the tension of the string, $\angle ESA = \angle POM = \theta$, $\phi =$ the whole angle through which the cylinder has revolved

about its centre of gravity; also, let m = the mass of the cylinder, k = its radius of gyration about its axis, α = the inclination of the plane to the horizon, and $AE = MO = a$.

Then for the motion of the cylinder we have, resolving forces parallel to the plane,

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - T\cos\theta \ldots\ldots\ldots\ldots (1);$$

and taking moments about O, the centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Ta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2):$$

by the elimination of T between these two equations, we get

$$a\frac{d^2x}{dt^2} = ag\sin\alpha - k^2\cos\theta\frac{d^2\phi}{dt^2} \ldots\ldots\ldots\ldots (3).$$

Take along EA, produced if necessary, Ep equal to EP: then, if the cylinder were made to roll from E to p, and then Ep were made to revolve about E into the position EP, the cylinder would clearly on the whole have revolved about its centre of gravity through the very angle which actually belongs to its real motion in setting free the length EP of the string. Now, in the first stage of the hypothetical motion, the cylinder will obviously move through an angle equal to $\frac{Ep}{a}$ or $\frac{EP}{a}$, which is equal to $\cot\theta$; and, in the second stage, through an angle $pES = \frac{1}{2}\pi - \theta$, in an opposite direction. Hence clearly we have

$$\phi = \cot\theta - (\tfrac{1}{2}\pi - \theta) = \cot\theta + \theta - \tfrac{1}{2}\pi \ldots\ldots\ldots\ldots (4).$$

Also, from the geometry, it is obvious that

$$x = \frac{a}{\sin\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (4) and (5) we have

$$d\phi = -\frac{d\theta}{\sin^2\theta} + d\theta = -\frac{\cos^2\theta}{\sin^2\theta}d\theta \ldots\ldots\ldots\ldots (6),$$

$$dx = -\frac{a\cos\theta}{\sin^2\theta}d\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Multiplying (3) by $2\dfrac{dx}{dt}$, we get

$$2a\frac{dx}{dt}\frac{d^2x}{dt^2} = 2ag\sin\alpha\frac{dx}{dt} - 2k^2\frac{dx}{dt}\cos\theta\frac{d^2\phi}{dt^2};$$

but from (6) and (7) it is clear that

$$\cos\theta\frac{dx}{dt} = a\frac{d\phi}{dt} \quad\text{.........................}\quad (8);$$

hence we obtain

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2g\sin\alpha\frac{dx}{dt} - 2k^2\frac{d\phi}{dt}\frac{d^2\phi}{dt^2};$$

integrating, and adding the arbitrary constant C,

$$\frac{dx^2}{dt^2} + k^2\frac{d\phi^2}{dt^2} = 2gx\sin\alpha + C;$$

but, initially, $\dfrac{dx}{dt} = 0$, $\dfrac{d\phi}{dt} = 0$, $x = a$; hence

$$0 = 2ga\sin\alpha + C,$$

and therefore

$$\frac{dx^2}{dt^2} + k^2\frac{d\phi^2}{dt^2} = 2g\sin\alpha\,(x - a);$$

substituting in this equation the values of x, $d\phi$, dx, given in (5), (6), (7), we have

$$\left(\frac{a^2\cos^2\theta}{\sin^4\theta} + \frac{k^2\cos^4\theta}{\sin^4\theta}\right)\frac{d\theta^2}{dt^2} = 2g\sin\alpha\left(\frac{a}{\sin\theta} - a\right),$$

and therefore

$$\frac{d\theta^2}{dt^2} = \frac{2ga\sin\alpha\,(1-\sin\theta)\sin^3\theta}{\cos^2\theta\,(a^2 + k^2\cos^2\theta)} \quad\text{.................}\quad (9);$$

which gives the angular velocity of the string about E in terms of its inclination to the plane, or for any position of the cylinder.

Again, from (8), we have

$$a\frac{d^2\phi}{dt^2} = \cos\theta\frac{d^2x}{dt^2} - \sin\theta\frac{dx}{dt}\frac{d\theta}{dt},$$

and therefore

$$\frac{d^2x}{dt^2} = \frac{a}{\cos\theta}\frac{d^2\phi}{dt^2} + \tan\theta\frac{dx}{dt}\frac{d\theta}{dt}$$

$$= \frac{a}{\cos\theta}\frac{d^2\phi}{dt^2} - \frac{a}{\sin\theta}\frac{d\theta^2}{dt^2}, \text{ by (7)};$$

substituting this value of $\frac{d^2x}{dt^2}$ in (3), we obtain

$$\left(\frac{a}{\cos\theta}+\frac{k^2}{a}\cos\theta\right)\frac{d^2\phi}{dt^2}=g\sin\alpha+\frac{a}{\sin\theta}\frac{d\theta^2}{dt^2},$$

and therefore, by (9),

$$\frac{a^2+k^2\cos^2\theta}{a\cos\theta}\frac{d^2\phi}{dt^2}=g\sin\alpha+\frac{2ga^2\sin\alpha\sin^2\theta\,(1-\sin\theta)}{\cos^2\theta\,(a^2+k^2\cos^2\theta)}$$

$$=g\sin\alpha\frac{a^2(1+\sin^2\theta-2\sin^3\theta)+k^2\cos^4\theta}{\cos^2\theta\,(a^2+k^2\cos^2\theta)};$$

hence, by (2), we have for the tension of the string for any position of the cylinder,

$$T=\frac{mk^2}{a}\frac{d^2\phi}{dt^2}=mk^2g\sin\alpha\,\frac{a^2(1+\sin^2\theta-2\sin^3\theta)+k^2\cos^4\theta}{\cos\theta\,(a^2+k^2\cos^2\theta)^2}.$$

Euler; *Nova Acta Acad. Petrop.* 1795; p. 64.

(6) A uniform heavy rod OA, (fig. 195), which is at liberty to oscillate in a vertical plane about a horizontal axis through O, falls from a horizontal position; to determine the angle included between the direction of the rod and the direction of the pressure upon the fixed axis, for any position of the rod.

From O draw Om at right angles to OA and to the fixed axis; and produce AO indefinitely to a point n. Let R, S, denote the resolved parts of the reaction of the fixed axis along Om, On, for any position of the rod. Draw Ox horizontal and at right angles to the fixed axis. Let $OA=a$; $m=$ the mass of the rod; $\angle AOx=\theta$, at any time t. Then, for the motion of the rod about its centre of gravity G, the moment of inertia about G being $\frac{1}{12}ma^2$,

$$\frac{1}{12}ma^2\frac{d^2\theta}{dt^2}=\frac{1}{2}aR,$$

$$ma\frac{d^2\theta}{dt^2}=6R\;\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

Also, for the motion about O, the moment of inertia about O being $\frac{1}{3}ma^2$,

$$\frac{1}{3}ma^2\frac{d^2\theta}{dt^2}=mg\,.\,\tfrac{1}{2}a\cos\theta,$$

$$2a\frac{d^2\theta}{dt^2}=3g\cos\theta\;\ldots\ldots\ldots\ldots\ldots\ldots\;(2).$$

Eliminating $\frac{d^2\theta}{dt^2}$ between (1) and (2), we get

$$R = \tfrac{1}{4} mg \cos\theta \text{ .... } (3).$$

Again, equating S to the resolved part of the weight along OA and the centrifugal force,

$$S = mg \sin\theta + \int_0^a \left(\frac{mdr}{a} r \frac{d\theta^2}{dt^2}\right)$$

$$= mg \sin\theta + \tfrac{1}{2} ma \frac{d\theta^2}{dt^2} \text{ .... } (4).$$

Again, multiplying (2) by $\frac{d\theta}{dt}$, and integrating,

$$a \frac{d\theta^2}{dt^2} = C + 3g \sin\theta;$$

but $\frac{d\theta}{dt} = 0$, when $\theta = 0$; hence $C = 0$, and therefore

$$a \frac{d\theta^2}{dt^2} = 3g \sin\theta;$$

hence, from (4), we get

$$S = mg \sin\theta + \tfrac{3}{2} mg \sin\theta = \tfrac{5}{2} mg \sin\theta \text{ .... } (5).$$

Let ϕ be the angle which the whole reaction of the fixed axis makes with the line On; then

$$\tan\phi = \frac{R}{S},$$

and therefore, by the equations (3) and (5),

$$\tan\theta \tan\phi = \tfrac{1}{10},$$

which gives the value of ϕ for any position of the rod: ϕ is evidently the angle between the direction of the whole pressure on the fixed axis and the length OA of the rod.

A solution of this problem was given in Chap. VI., by the direct application of D'Alembert's Principle.

(7) A uniform rod, acted on by gravity, is oscillating in a vertical plane about one extremity; to find the tendency of the vis inertia in any position to bend the rod at any point, and to ascertain the point at which this tendency is a maximum.

Let OA (fig. 196) be the position of the rod at any time t; Ox an indefinite horizontal line through O, the fixed extremity of the rod, in the vertical plane through OA. Take C any

point in OA, P any point in CA. Let $OA=2a$, $OC=c$, $OP=r$, $\angle AOx=\theta$, $k=$ the radius of gyration about O; $m=$ the mass of the rod.

Then the *force gained* by an element dr of the rod at the point P, resolved at right angles to OP, will be equal to

$$m\frac{dr}{2a}\left(r\frac{d^2\theta}{dt^2}-g\cos\theta\right);$$

and the moment of this about C will be equal to

$$\frac{m}{2a}\left(r\frac{d^2\theta}{dt^2}-g\cos\theta\right)(r-c)\,dr;$$

hence the whole moment to produce bending at C will be equal to

$$\frac{m}{2a}\int_c^{2a}\left(r\frac{d^2\theta}{dt^2}-g\cos\theta\right)(r-c)\,dr \;\ldots\ldots\ldots\ldots\; (1).$$

But, for the motion of the rod, we have

$$mk^2\frac{d^2\theta}{dt^2}=mga\cos\theta,$$

and therefore, $\frac{4}{3}a^2$ being the value of k^2,

$$\frac{d^2\theta}{dt^2}=\frac{3g}{4a}\cos\theta.$$

Hence the expression (1) becomes

$$\frac{mg\cos\theta}{8a^2}\int_c^{2a}(3r-4a)(r-c)\,dr$$

$$=\frac{mg\cos\theta}{8a^2}\int_c^{2a}\{3(r-c)-(4a-3c)\}(r-c)\,dr$$

$$=\frac{mg\cos\theta}{8a^2}\{(2a-c)^3-\tfrac{1}{2}(4a-3c)(2a-c)^2\}$$

$$=\frac{mg\cos\theta}{8a^2}(2a-c)^2\{2a-c-\tfrac{1}{2}(4a-3c)\}$$

$$=\frac{mg\cos\theta}{16a^2}c(2a-c)^2.$$

When this expression is a maximum, we have

$$(2a-c)^2-2c(2a-c)=0,$$

$$2a-3c=0,\quad c=\tfrac{2}{3}a,$$

or

$$OC=\tfrac{1}{3}OA.$$

The following is a different solution of the same problem.

Let X, Y, (fig. 197), be the transversal and longitudinal actions and reactions of any two portions OC, CA, of the rod; and let μ be the wrenching force at C estimated as tending to elevate OC.

Then, for the motion of OC, taking moments about O,

$$m\frac{c}{2a}\cdot\tfrac{1}{3}c^2\frac{d^2\theta}{dt^2} = m\frac{c}{2a}\cdot\tfrac{1}{2}gc\cos\theta - X.c - \mu,$$

and, for the motion of CA, taking moments about its centre of gravity,

$$m\frac{2a-c}{2a}\cdot\tfrac{1}{3}\left(\frac{2a-c}{2}\right)^2\cdot\frac{d^2\theta}{dt^2} = \mu - X\frac{2a-c}{2}.$$

But, for the motion of the whole rod,

$$\frac{d^2\theta}{dt^2} = \frac{3g}{4a}\cos\theta:$$

hence the equations for the motion of the two pieces become

$$\mu + cX = \frac{mgc^2\cos\theta}{8a^2}(2a-c),$$

and $$2\mu - (2a-c)X = \frac{mg(2a-c)^3\cos\theta}{32a^2}.$$

Eliminating X, we shall easily see that

$$\mu = \frac{mgc(2a-c)^2\cos\theta}{16a^2},$$

the same expression for the wrench as we obtained in the former solution.

(8) A uniform beam is supported symmetrically on two props: to find where they must be placed in order that, when one of them is removed, the instantaneous pressure on the other may be the same as the previous statical pressure.

Let A, fig. (198), be the position of the prop which is not removed; G the centre of gravity of the beam. Let $AG = h$, k = the radius of gyration of the beam about G, m = the mass of

the beam, R = the reaction of the prop at A before and immediately after the removal of the other prop, f = the instantaneous angular acceleration.

Then, taking moments about A,

$$m(h^2+k^2)f = mgh \quad \text{.....................} \quad (1).$$

Again, taking moments about G,

$$mk^2f = Rh \quad \text{............................} \quad (2).$$

Also, for the equilibrium of the beam while supported by both props,

$$R = \tfrac{1}{2} mg \quad \text{..............................} \quad (3).$$

From (1), (2), (3), we see that $h = k$; and therefore $2k$ is the required distance between the two props.

(9) A hemisphere revolves about an axis, which coincides with a diameter of its base, and is inclined at a given angle to the vertical, from a position of instantaneous rest in which the plane containing the centre of gravity and fixed axis was perpendicular to the vertical plane through that axis: to find the whole pressure on the axis, when these two planes coincide.

Let ACB, fig. (199), be the axis of revolution, Ax a vertical line, G the centre of gravity of the hemisphere in any position during the motion, H the lowest position of G. Let $\angle BAx = \alpha$, a = the radius of the sphere, $CG = c$, $\angle DCE = \theta$, m = the mass of the hemisphere.

The whole pressure on the axis, when G is at H, will be equal to the sum of the pressure due to gravity, and the pressure due to centrifugal force.

The weight of the hemisphere may be resolved into $mg\cos\alpha$, parallel to BA, which produces no effect on the motion, and $mg\sin\alpha$, parallel to CH. The moment of the latter component about AB is equal to

$$mg\sin\alpha \,.\, c\sin\theta:$$

hence
$$mk^2\frac{d^2\theta}{dt^2} = -mgc\sin\alpha\sin\theta,$$

and therefore, since $\dfrac{d\theta}{dt} = 0$ when $\theta = \frac{1}{2}\pi$,

$$\frac{d\theta^2}{dt^2} = \frac{2gc}{k^2} \sin \alpha \cos \theta,$$

or, since $c = \frac{3}{8} a$, and $k^2 = \frac{2}{5} a^2$,

$$\frac{d\theta^2}{dt^2} = \frac{15g}{8a} \sin \alpha \cos \theta.$$

Hence the pressure on the axis, arising from centrifugal force, is equal to

$$m \frac{d\theta^2}{dt^2} \cdot CH = \frac{45}{64} mg \sin \alpha.$$

Again, the pressure arising from gravity is equivalent to $mg \sin \alpha$, at right angles to BA, and $mg \cos \alpha$, parallel to BA.

Hence the whole pressure exerted on the fixed axis at right angles to it is equal to

$$mg \sin \alpha \left(1 + \frac{45}{64}\right) = \frac{109}{64} mg \sin \alpha.$$

The resultant of the two pressures on the fixed axis is therefore equal to

$$mg \cdot \left\{\cos^2 \alpha + \left(\frac{109}{64}\right)^2 \sin^2 \alpha\right\}^{\frac{1}{2}}.$$

(10) A rigid body is in motion, a point of the body being fixed and no forces acting upon it: to determine the relation between the moments of inertia about its principal axes at the fixed point in order that the angular velocity of the body about its instantaneous axis may be constant.

If A, B, C, be the moments of inertia about the three principal axes, and ω_1, ω_2, ω_3, the angular velocities about these axes, then, by the general formulæ of rotation, no forces acting,

$$A \frac{d\omega_1}{dt} = (B - C) \omega_2 \omega_3, \qquad B \frac{d\omega_2}{dt} = (C - A) \omega_3 \omega_1,$$

$$C \frac{d\omega_3}{dt} = (A - B) \omega_1 \omega_2.$$

Also, ω being the angular velocity about the instantaneous axis,

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2,$$

and therefore, ω being supposed to be constant,

$$\omega_1 \frac{d\omega_1}{dt} + \omega_2 \frac{d\omega_2}{dt} + \omega_3 \frac{d\omega_3}{dt} = 0.$$

Eliminating the quantities

$$\frac{d\omega_1}{dt}, \frac{d\omega_2}{dt}, \frac{d\omega_3}{dt},$$

between the equations involving them, we get, as the required relation,

$$\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C} = 0.$$

(11) A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane: to find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

The required angle of depression is equal to

$$\alpha - \sin^{-1}\left(\frac{14}{15} \sin \alpha\right).$$

(12) A uniform rod is placed in a given position with its lower end upon a smooth horizontal plane: supposing a horizontal force to be continually applied at its lower end such as to cause the rod to descend in a vertical plane with a given uniform angular velocity, to find the velocity of the lower end of the rod in any position.

If $\omega =$ the angular velocity, $\theta =$ the inclination of the rod at any time to the horizon, and $\alpha =$ the initial value of θ; the velocity of the lower end will be equal to

$$\frac{g}{\omega} \log \left(\frac{\sin \alpha}{\sin \theta}\right).$$

If $\omega = 0$, the lower end will have traversed a space equal to $\frac{1}{2} g t^2 \cot \alpha$ at the end of a time t.

(13) A heavy rod is suspended from a fixed point by two inextensible strings without weight, the strings and the rod

forming an equilateral triangle; supposing either of the strings to be cut, to determine the initial tension of the other.

The required tension is equal to

$$\frac{\sqrt{12}}{11}.W,$$

where W is the weight of the rod.

(14) A uniform sphere, moveable about a fixed point in its surface, rests against an inclined plane: supposing the diameter which passes through the fixed point to be horizontal, to determine whether, if the plane be suddenly removed, the pressure on the fixed point will be increased or diminished.

The pressure will be increased or diminished accordingly as the inclination of the plane was less or greater than $\tan^{-1}\frac{2}{7}$.

(15) A hemisphere oscillates about a horizontal axis which coincides with a diameter of the base: to compare the maximum pressure on the axis with the weight of the hemisphere, the base of the hemisphere at the commencement of the motion being inclined to the horizon at an angle of $60°$.

The greatest pressure $= \frac{173}{128} \times$ weight of hemisphere.

(16) A cone, moveable about a fixed horizontal diameter of its base, is supported with its axis horizontal by a vertical string fastened to its vertex: supposing the string to be cut, to compare the initial pressure on the fixed diameter with the pressure in the former case.

If α be the vertical angle of the cone, P the pressure on the horizontal diameter before the string is cut, and P' the pressure after it is cut, then

$$P' = P.\frac{5 - 3\cos\alpha}{5 - \cos\alpha}.$$

(17) A homogeneous sphere is suspended by a fine wire attached to a fixed point at its upper extremity: the sphere is then turned round by the hand through n revolutions, and then let go: to determine the motion communicated to it by the

untwisting of the wire, the elasticity of torsion being supposed proportional to the angle.

If θ be the trigonometrical angle through which, at the end of any time t, the sphere has been twisted from its position of rest; then, ρ denoting the density of the sphere and μ a constant, the whole motion is expressed by the equation

$$\theta = 2n\pi \cos \left\{ \left(\frac{15\mu}{8\pi\rho a^5} \right)^{\frac{1}{2}} t \right\}.$$

(18) A uniform rod, not acted on by any forces, is in motion, its ends being constrained to slide along two fixed rods at right angles to each other in one plane: to find the wrenching force at any point.

Let AB be the rod, C any point in it, O the intersection of the two fixed rods; let CH, CK, be perpendiculars from C upon OA, OB, respectively; let m = the mass of AB. Then the angular velocity ω of AB will be invariable, and the wrenching force at C will be equal to

$$\tfrac{1}{2} m\omega^2 . CH . CK.$$

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

(19) An angular velocity having been impressed upon a heterogeneous sphere, about an axis, perpendicular to the vertical plane which contains its centre of gravity G and its geometrical centre C, and passing through G (fig. 192), it is then placed upon a smooth horizontal plane; to determine the magnitude of the impressed angular velocity that G may rise into a point in the vertical line SCK through C, and there rest; the initial magnitude of the angle between CG and the vertical radius CS being given.

Let $CG = c$, k = the radius of gyration about G, α = the initial value of the angle GCS, and ω = the required angular velocity; then ω will be determined by the equation

$$(k^2 + c^2 \sin^2 \alpha)\, \omega^2 = 2cg\, (1 + \cos \alpha).$$

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(20) A chain, ten yards long, consisting of indefinitely small equal links, being laid straight on a perfectly smooth horizontal plane, except one part, a yard in length, which hangs down perpendicularly below the plane; in what time will the chain entirely quit the plane?

The time $= 2.890663$ seconds nearly.

Lady's and Gentleman's Diary, 1758; *Diarian Repository*, p. 683.

SECT. 2. *Several Bodies.*

(1) A wheel and axle is loaded with given weights P and Q, (fig. 200), which are not in equilibrium; to determine their motion and the tension of the strings by which the weights are suspended.

Through C, the centre of the wheel and axle, draw the horizontal line ACB meeting the strings in A and B; let $AC = a$, $BC = a'$; $m =$ the mass of P, $m' =$ that of Q, $\mu =$ that of the wheel and axle together; $k =$ the radius of gyration of the wheel and axle about their common axis; $AP = x$, $BQ = x'$, $T =$ the tension of AP, $T' =$ the tension of BQ; $\theta =$ the angle through which the wheel and axle have revolved at the end of the time t about their common axis.

Then, for the motion of P, we have

$$m\frac{d^2x}{dt^2} = mg - T \text{.....................}(1);$$

for the motion of Q,

$$m'\frac{d^2x'}{dt^2} = m'g - T' \text{.......................}(2);$$

and, for the rotation of the wheel and axle,

$$\mu k^2\frac{d^2\theta}{dt^2} = Ta - T'a' \text{.....................}(3).$$

But, from the geometry, it is clear that

$$\frac{dx}{dt} = a\frac{d\theta}{dt}, \quad \frac{dx'}{dt} = -a'\frac{d\theta}{dt};$$

hence, from (1) and (2),

$$ma\frac{d^2\theta}{dt^2} = mg - T \quad\text{.......................... (4)},$$

$$-m'a'\frac{d^2\theta}{dt^2} = m'g - T' \quad\text{.......................... (5)}.$$

Substituting the values of T and T' from (4) and (5) in the equation (3), we get

$$\mu k^2\frac{d^2\theta}{dt^2} = ma\left(g - a\frac{d^2\theta}{dt^2}\right) - m'a'\left(g + a'\frac{d^2\theta}{dt^2}\right),$$

$$(ma^2 + m'a'^2 + \mu k^2)\frac{d^2\theta}{dt^2} = g\,(ma - m'a') \quad\text{............ (6)};$$

whence θ is immediately obtained in terms of t, the initial values of θ and $\frac{d\theta}{dt}$ being supposed to be known.

From (4) and (6) we have

$$T = mg - \frac{mag\,(ma - m'a')}{ma^2 + m'a'^2 + \mu k^2};$$

and, from (5), (6),

$$T' = m'g + \frac{m'a'g\,(ma - m'a')}{ma^2 + m'a'^2 + \mu k^2}.$$

(2) Two equal uniform rods AC, BC, (fig. 201), having a compass joint at C, are laid in a line upon a horizontal plane. A string CDP having a given weight P at one end passes over a smooth pin D above the plane, and has its other end fastened to C which is vertically beneath the pin; to determine the motion when P descends.

Let $AC = 2a = BC$, $\angle CAB = \theta$; R = the vertical reaction of the plane at each of the points A and B; T = the tension of the string; S = the mutual action of the two rods at the joint, which will evidently take place in a horizontal line parallel to AB; m = the mass of each of the rods, μ = the mass of the weight P. Let G be the centre of gravity of the rod AC; draw GH, CE, at right angles to AB; let $EH = x$, $GH = y$, k = the radius of gyration of AC about G.

Then, for the motion of the rod AC, we have, resolving forces horizontally,

$$m\frac{d^2x}{dt^2} = S \text{..........................} (1);$$

resolving vertically, $\frac{1}{2}T$ being the force exerted by the string on each rod,

$$m\frac{d^2y}{dt^2} = R + \tfrac{1}{2}T - mg \text{..................} (2);$$

and taking moments about G,

$$mk^2\frac{d^2\theta}{dt^2} = Sa\sin\theta + \tfrac{1}{2}Ta\cos\theta - Ra\cos\theta \text{............} (3).$$

Also, for the motion of P, the increment of DP being double that of GH,

$$2\mu\frac{d^2y}{dt^2} = \mu g - T \text{......................} (4).$$

Multiplying the equation (2) by $a\cos\theta$, we have

$$ma\cos\theta\left(\frac{d^2y}{dt^2} + g\right) = (R + \tfrac{1}{2}T)\,a\cos\theta,$$

and therefore, adding this equation to the equation (3),

$$mk^2\frac{d^2\theta}{dt^2} + ma\cos\theta\left(\frac{d^2y}{dt^2} + g\right) = Sa\sin\theta + Ta\cos\theta;$$

hence, from (1) and (4),

$$mk^2\frac{d^2\theta}{dt^2} + ma\cos\theta\left(\frac{d^2y}{dt^2} + g\right) = ma\sin\theta\frac{d^2x}{dt^2} + \mu a\cos\theta\left(g - 2\frac{d^2y}{dt^2}\right);$$

and therefore, since $x = a\cos\theta$, and $y = a\sin\theta$,

$$mk^2\frac{d^2\theta}{dt^2} + ma^2\left(\cos\theta\frac{d^2\sin\theta}{dt^2} - \sin\theta\frac{d^2\cos\theta}{dt^2}\right) + 2\mu a^2\cos\theta\frac{d^2\sin\theta}{dt^2}$$
$$= ag(\mu - m)\cos\theta,$$

$$(ma^2 + mk^2)\frac{d^2\theta}{dt^2} + 2\mu a^2\cos\theta\frac{d^2\sin\theta}{dt^2} = ag(\mu - m)\cos\theta.$$

Multiplying both sides of this equation by $2\frac{d\theta}{dt}$, and integrating, we have, since $\frac{d\theta}{dt} = 0$ when $\theta = 0$,

$$(ma^2 + mk^2 + 2\mu a^2\cos^2\theta)\frac{d\theta^2}{dt^2} = 2ag(\mu - m)\sin\theta,$$

which determines the angular velocity of the rods for any position.

The value of $\frac{d\theta}{dt}$ and therefore of $\frac{d^2\theta}{dt^2}$ being known in terms of θ, we may readily obtain the values of R, S, and T, from the equations (1), (2), (4), in terms of the same angle.

(3) A tube, moveable in a horizontal plane about a vertical axis, is charged with any number of balls at assigned intervals; supposing a given angular velocity to be communicated to the tube, it is required to determine the motion of the tube and of the balls.

Let a, a', a'',...be the initial distances of the balls from the fixed axis, and r, r', r'',... their distances at any time t from the commencement of the motion. Let m, m', m'',... be the masses of the balls, μ of the tube; and let θ be the angle through which the tube has revolved at the end of the time t. Let R, R', R'',... denote the mutual actions and reactions of the balls and the tube. Then, μk^2 denoting the moment of inertia of the tube about the vertical axis, we shall have, for the motion of the tube,

$$\mu k^2 \frac{d^2\theta}{dt^2} = Rr + R'r' + R''r'' + \dots\dots\dots\dots \text{(1)}.$$

Also, for the motion of the balls, m, m', m'',...... we have

$$m\frac{d^2x}{dt^2} = R\frac{y}{r}, \quad m\frac{d^2y}{dt^2} = -R\frac{x}{r} \dots\dots\dots\dots \text{(2)},$$

$$m'\frac{d^2x'}{dt^2} = R'\frac{y'}{r'}, \quad m'\frac{d^2y'}{dt^2} = -R'\frac{x'}{r'} \dots\dots\dots\dots \text{(3)},$$

$$m''\frac{d^2x''}{dt^2} = R''\frac{y''}{r''}, \quad m''\frac{d^2y''}{dt^2} = -R''\frac{x''}{r''} \dots\dots\dots\dots \text{(4)},$$

. . . .

. . . .

where (x, y), (x', y'), (x'', y''),... are the rectangular co-ordinates of the balls at the time t.

Multiplying the former and the latter of the equations (2) by y and x respectively, and subtracting the latter from the former of the resulting equations, we get

$$m\left(y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}\right)=Rr;$$

and therefore, since $x=r\cos\theta$, $y=r\sin\theta$, the axis of x being supposed to coincide with the initial position of the tube, we may readily obtain, by substitution,

$$m\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-Rr.$$

In like manner, from the equations of (3), (4),.... we may get

$$m'\frac{d}{dt}\left(r'^2\frac{d\theta}{dt}\right)=-R'r',$$

$$m''\frac{d}{dt}\left(r''^2\frac{d\theta}{dt}\right)=-R''r'',$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Hence from (1) we have

$$\mu k^2\frac{d^2\theta}{dt^2}=-\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}+m'r'^2\frac{d\theta}{dt}+m''r''^2\frac{d\theta}{dt}+\ldots.\right);$$

integrating, we get

$$\mu k^2\frac{d\theta}{dt}=C-(mr^2+m'r'^2+m''r''^2+\ldots\ldots)\frac{d\theta}{dt};$$

but, supposing ω to be the initial value of $\frac{d\theta}{dt}$, we have

$$\mu k^2\omega=C-(ma^2+m'a'^2+m''a''^2+\ldots)\,\omega;$$

hence $$\frac{d\theta}{dt}=\frac{\mu k^2+ma^2+m'a'^2+m''a''^2+\ldots}{\mu k^2+mr^2+m'r'^2+m''r''^2+\ldots}\,\omega \;\ldots\ldots\ldots\ldots (5).$$

Again, from the equations (2), we have

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}=0,$$

and thence, substituting for x and y their values in r and θ,

$$\frac{d^2r}{dt^2}=r\frac{d\theta^2}{dt^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In the same way, from (3), we may get

$$\frac{d^2r'}{dt^2}=r'\frac{d\theta^2}{dt^2};$$

and therefore, eliminating $\frac{d\theta}{dt}$ between these two equations,

$$r'\frac{d^2r}{dt^2}=r\frac{d^2r'}{dt^2};$$

integrating and bearing in mind that both $\frac{dr}{dt}$ and $\frac{dr'}{dt}$ are initially equal to zero,

$$r'\frac{dr}{dt}=r\frac{dr'}{dt},$$

and therefore
$$\frac{dr}{r}=\frac{dr'}{r'};$$

integrating again we have, a, a', being the initial values of r, r',

$$\frac{r}{a}=\frac{r'}{a'},\quad r'=\frac{a'}{a}r.$$

In precisely the same way it may be shewn that

$$r''=\frac{a'}{a}r,\quad r'''=\frac{a'''}{a}r,\ \ldots\ldots$$

Hence from (5) we have

$$\frac{d\theta}{dt}=\frac{\mu k^2+ma^2+m'a'^2+m''a''^2+\ldots\ldots\ldots}{\mu k^2+\left(ma^2+m'a'^2+m''a''^2+\ldots\right)\frac{r^2}{a^2}}\,\omega\ldots\ldots\ldots\ldots\ldots(7).$$

From (6) and (7) we obtain

$$\frac{d^2r}{dt^2}=\omega^2r\left\{\frac{\mu k^2+ma^2+m'a'^2+m''a''^2+\ldots\ldots\ldots}{\mu k^2+\left(ma^2+m'a'^2+m''a''^2+\ldots\right)\frac{r^2}{a^2}}\right\}^2.$$

Multiplying both sides of this equation by $2\frac{dr}{dt}$, integrating, and bearing in mind that $\frac{dr}{dt}=0$ when $r=a$, we shall easily see that

$$\frac{dr^2}{dt^2}=\omega^2(r^2-a^2)\frac{\mu k^2+ma^2+m'a'^2+m''a''^2+\ldots\ldots.}{\mu k^2+(ma^2+ma'^2+m''a''^2+\ldots)\frac{r^2}{a^2}}\ldots\ldots\ldots\ldots(8).$$

The equations (7) and (8) will give us, for any assigned distance of the ball m from the axis of rotation, the angular

velocity of the tube and the velocity of the ball m within it. If between (7) and (8) we eliminate dt, we shall obtain the differential equation in polar co-ordinates to the path of m in the horizontal plane passing through the axis of the tube. Similar results may evidently be obtained for the other balls with which the tube is charged.

COR. If $\mu = 0$, the equations (7) and (8) become

$$\frac{d\theta}{dt} = \frac{a^2\omega}{r^2}, \qquad \frac{dr^2}{dt^2} = (r^2 - a^2)\frac{a^2\omega^2}{r^2},$$

and therefore, eliminating dt,

$$\frac{dr^2}{d\theta^2} = (r^2 - a^2)\frac{r^2}{a^2}, \qquad d\theta = \frac{adr}{r(r^2 - a^2)^{\frac{1}{2}}} = -\frac{d\frac{a}{r}}{\left(1 - \frac{a^2}{r^2}\right)^{\frac{1}{2}}}:$$

integrating, and remembering that $\theta = 0$ when $r = a$,

$$\theta = \cos^{-1}\frac{a}{r}, \qquad \frac{a}{r} = \cos\theta.$$

Again, to determine the relation between r and t, we have

$$dt = \frac{1}{\omega a}\frac{rdr}{(r^2 - a^2)^{\frac{1}{2}}}, \qquad t = \frac{1}{\omega a}(r^2 - a^2)^{\frac{1}{2}},$$

and therefore $$\frac{r^2}{a^2} = 1 + \omega^2 t^2.$$

Similar relations holding good for the other balls, we have, for the equations to their paths,

$$\frac{a}{r} = \frac{a'}{r'} = \frac{a''}{r''} = \ldots\ldots = \cos\theta;$$

which shew that they all move in straight lines at right angles to the initial position of the tube; and for their distances from the axis of rotation at any time,

$$\frac{r^2}{a^2} = \frac{r'^2}{a'^2} = \frac{r''^2}{a''^2} = \ldots\ldots = 1 + \omega^2 t^2.$$

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1742, p. 48. Daniel Bernoulli; *Mém. de l'Acad. des Sciences de Berlin*, 1745, p. 54. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 71.

(4) A heavy particle P descends down a smooth inclined plane BA, (fig. 202), forming the upper surface of a solid BAC, which is capable of sliding freely along a smooth horizontal plane OAx; to determine the motion of the particle and of the body, both of which are supposed to have initially no motion.

Let PM be at right angles to Ox, and let B be the point in the inclined plane which the particle occupies initially; let A be supposed to coincide with O at the commencement of the motion. Let $OM = x$, $PM = y$, $OA = s$, $AB = a$, $BP = s'$, $\angle BAC = \alpha$; and R = the action and reaction of the plane and the particle. Then, m denoting the mass of the particle and m' of the body, we shall have

$$m\frac{d^2y}{dt^2} = R\cos\alpha - mg \quad\ldots\ldots\ldots\ldots\ldots\ (1),$$

$$m\frac{d^2x}{dt^2} = -R\sin\alpha \quad\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$m'\frac{d^2s}{dt^2} = R\sin\alpha \quad\ldots\ldots\ldots\ldots\ldots\ (3).$$

But $y = (a - s')\sin\alpha$, $x = s + (a - s')\cos\alpha$; hence, from (1),

$$m\sin\alpha\frac{d^2s'}{dt^2} = mg - R\cos\alpha \quad\ldots\ldots\ldots\ldots\ldots\ (4),$$

and, from (2),

$$m\frac{d^2s}{dt^2} - m\cos\alpha\frac{d^2s'}{dt^2} = -R\sin\alpha \quad\ldots\ldots\ldots\ldots\ldots\ (5).$$

Adding together (3) and (5), we have

$$(m + m')\frac{d^2s}{dt^2} - m\cos\alpha\frac{d^2s'}{dt^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ (6).$$

Multiplying (3) by $\cos\alpha$ and (4) by $\sin\alpha$, we have, adding together the resulting equations,

$$m'\cos\alpha\frac{d^2s}{dt^2} + m\sin^2\alpha\frac{d^2s'}{dt^2} = mg\sin\alpha \quad\ldots\ldots\ldots\ldots\ (7).$$

Multiplying (6) by $\sin^2\alpha$, (7) by $\cos\alpha$, and adding together the resulting equations,

$$(m\sin^2\alpha + m')\frac{d^2s}{dt^2} = mg\sin\alpha\cos\alpha;$$

integrating twice with respect to t, and bearing in mind that $s=0$ and $\frac{ds}{dt}=0$ when $t=0$, we obtain

$$s = \tfrac{1}{2}gt^2 \frac{m \sin\alpha \cos\alpha}{m \sin^2\alpha + m'} \quad \text{.....................} \quad (8).$$

Again, multiplying (7) by $m+m'$, (6) by $m' \cos\alpha$, and subtracting the latter of the resulting equations from the former, we have

$$m\,(m \sin^2\alpha + m')\frac{d^2s'}{dt^2} = mg \sin\alpha\,(m+m'),$$

$$\frac{d^2s'}{dt^2} = \frac{g \sin\alpha\,(m+m')}{m \sin^2\alpha + m'};$$

integrating twice, and recollecting that $s'=0$ and $\frac{ds'}{dt}=0$ when $t=0$, we get

$$s' = \tfrac{1}{2}gt^2 \frac{(m+m') \sin\alpha}{m \sin^2\alpha + m'} \quad \text{.....................} \quad (9).$$

The equation (8) gives the position of the moveable inclined plane, and (9) the place of the particle on the plane at any time.

Again, by (3),

$$R = \frac{m'}{\sin\alpha}\frac{d^2s}{dt^2} = \frac{mm'g \cos\alpha}{m \sin^2\alpha + m'},$$

which gives the value of the mutual pressure of the particle and the plane; the value of which, therefore, is invariable.

John Bernoulli; *Comment. Acad. Petrop.* 1730, p. 11. *Opera*, Tom. III. p. 365. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 28.

(5) A heavy particle is placed within a thin tube APB, (fig. 203), situated in a vertical plane, which passes through a horizontal line OE; the tube is attached rigidly to a body ABC, the lower surface of which is flat, and in contact with a smooth horizontal plane, along which it is able to slide freely; supposing the particle and the body to be initially at rest, to find their subsequent motions.

Let A be the point of the tube which the particle occupies initially, and O the initial position of the point B of the body; let

$OB = s$, the length AP of the tube $= s'$; $OM = x$, $PM = y$, where PM is vertical; ϕ = the inclination to the horizon of an element of the tube at P; m = the mass of the particle and m' = the mass of the body; R = the action and reaction of the tube and the particle. Then, for the motion of the particle, we have

$$m\frac{d^2x}{dt^2} = -R\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$m\frac{d^2y}{dt^2} = R\cos\phi - mg \ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, for the motion of the body,

$$m'\frac{d^2s}{dt^2} = R\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, from the geometry it is evident that

$$dx = ds - \cos\phi\, ds' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and $$dy = -\sin\phi\, ds' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

From (1) and (3) we have

$$m\frac{d^2x}{dt^2} + m'\frac{d^2s}{dt^2} = 0;$$

integrating, we get

$$m\frac{dx}{dt} + m'\frac{ds}{dt} = C;$$

where C is an arbitrary constant; and therefore, by (4),

$$(m+m')\frac{ds}{dt} - m\cos\phi\frac{ds'}{dt} = C;$$

but, by the conditions of the problem, $\frac{ds}{dt} = 0$, and $\frac{ds'}{dt} = 0$ simultaneously, and therefore $C = 0$; hence

$$(m+m')\frac{ds}{dt} - m\cos\phi\frac{ds'}{dt} = 0 \ldots\ldots\ldots\ldots(6).$$

Again, from (2) and (3),

$$m'\cos\phi\frac{d^2s}{dt^2} - m\sin\phi\frac{d^2y}{dt^2} = mg\sin\phi.$$

and therefore, by (5),

$$m'\cos\phi\frac{d^2s}{dt^2} + m\sin\phi\frac{d}{dt}\left(\sin\phi\frac{ds'}{dt}\right) = mg\sin\phi;$$

but, from (6), $\dfrac{d^2s}{dt^2} = \dfrac{m}{m+m'}\dfrac{d}{dt}\left(\cos\phi\dfrac{ds'}{dt}\right)$;

hence we have

$$\frac{m'}{m+m'}\cos\phi\frac{d}{dt}\left(\cos\phi\frac{ds'}{dt}\right)+\sin\phi\frac{d}{dt}\left(\sin\phi\frac{ds'}{dt}\right)=g\sin\phi.$$

Multiply both sides of this equation by $2\dfrac{ds'}{dt}$ and integrate; then

$$\frac{m'}{m+m'}\left(\cos\phi\frac{ds'}{dt}\right)^2+\left(\sin\phi\frac{ds'}{dt}\right)^2=2g\int\sin\phi\,ds',$$

$$(m\sin^2\phi+m')\frac{ds'^2}{dt^2}=2g(m+m')\int\sin\phi\,ds',$$

$$\frac{ds'}{dt}=\{2g(m+m')\}^{\frac{1}{2}}\left\{\frac{\int\sin\phi\,ds'}{m\sin^2\phi+m'}\right\}^{\frac{1}{2}}\text{............}(7);$$

from this equation, when the form of the tube and therefore the relation between ϕ and s' is known, we may determine the relation between s' and t.

Again, for the determination of the relation between s and t, we have, from (6) and (7),

$$\frac{ds}{dt}=\left(\frac{2g}{m+m'}\right)^{\frac{1}{2}}m\cos\phi\left\{\frac{\int\sin\phi\,ds'}{m\sin^2\phi+m'}\right\}^{\frac{1}{2}}:$$

the integral $\int\sin\phi\,ds'$, which enters into these formulæ, must evidently be so taken as to vanish when $s'=0$.

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1742, p. 41. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48.

(6) A smooth groove ALA', (fig. 204), is carved in a vertical plane in the body $ACBA'$, which is placed upon a smooth horizontal plane, along which it is able to slide freely; to find the form of the groove that a heavy particle may oscillate in it tautochronously, the time of an oscillation being given.

Let L be the lowest point of the groove, LM a vertical line through L, meeting a horizontal line PM through the place P of the particle at any time t; let $PM=x'$, $ML=y'$, arc $LP=s'$.

Then by the equation (7) of the preceding problem, putting $-ds'$ in place of ds', and retaining in other respects the same notation, we have

$$\frac{ds'}{dt} = -\{2g(m+m')\}^{\frac{1}{2}}\left\{\frac{-\int \sin\phi\, ds'}{m\sin^2\phi + m'}\right\}^{\frac{1}{2}};$$

but
$$-\int \sin\phi\, ds' = k - y',$$

if k be the initial value of y'; hence

$$\frac{ds'}{dt} = -\{2g(m+m')\}^{\frac{1}{2}}\left\{\frac{k-y'}{m\dfrac{dy'^2}{ds'^2}+m'}\right\},$$

and therefore, if τ be the time of half an oscillation,

$$\tau = -\frac{1}{\{2g(m+m')\}^{\frac{1}{2}}}\int_k^0 \frac{\dfrac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2}+m'\right)^{\frac{1}{2}}dy' \quad\text{.........(1)},$$

a quantity which, by the nature of the problem, must be independent of k: hence

$$\int \frac{\dfrac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2}+m'\right)^{\frac{1}{2}}dy'$$

must be of zero dimensions in y' and k; and therefore

$$\frac{\dfrac{ds'}{dy'}}{(k-y')^{\frac{1}{2}}}\left(m\frac{dy'^2}{ds'^2}+m'\right)^{\frac{1}{2}}$$

must evidently be of -1 dimensions in k and y': but it is clear that

$$\frac{ds'}{dy'}\left(m\frac{dy'^2}{ds'^2}+m'\right)^{\frac{1}{2}}$$

does not involve k; hence this expression must be of $-\frac{1}{2}$ dimensions in y', and therefore, α being some constant quantity,

$$\frac{ds'}{dy'}\left(m\frac{dy'^2}{ds'^2}+m'\right)^{\frac{1}{2}} = \frac{\alpha}{y'^{\frac{1}{2}}} \quad\text{..................(2)},$$

$$m + m' \frac{ds'^2}{dy'^2} = \frac{\alpha^2}{y'},$$

$$m' \frac{dx'^2}{dy'^2} = \frac{\alpha^2 - (m + m')\, y'}{y'},$$

$$\frac{dx'}{dy'} = \left(\frac{m + m'}{m'}\right)^{\frac{1}{2}} \left(\frac{2a - y'}{y'}\right)^{\frac{1}{2}},$$

where $2a = \dfrac{\alpha^2}{m + m'}$; integrating, we get for the equation to the curve,

$$x' = \left(\frac{m + m'}{m'}\right)^{\frac{1}{2}} \left\{(2ay' - y'^2)^{\frac{1}{2}} + a \text{ vers}^{-1} \frac{y'}{a}\right\} \ldots\ldots\ldots (3),$$

which is an elongated cycloid, which may be constructed from the ordinary cycloid by increasing the distance of each point of the curve from the axis in the ratio of $(m + m')^{\frac{1}{2}}$ to $m'^{\frac{1}{2}}$.

Again, from (1) and (2), we have

$$\tau = -\frac{\alpha}{\{2g\,(m + m')\}^{\frac{1}{2}}} \int_k^0 \frac{dy'}{(ky' - y'^2)^{\frac{1}{2}}}$$

$$= \frac{\pi \alpha}{\{2g\,(m + m')\}^{\frac{1}{2}}};$$

whence $\qquad \alpha^2 = \dfrac{2g\tau^2}{\pi^2}\,(m + m'),$

and therefore $\qquad 2a = \dfrac{\alpha^2}{m + m'} = \dfrac{2g\tau^2}{\pi^2}, \quad a = \dfrac{g\tau^2}{\pi^2}.$

Hence (3) may be written

$$x' = \left(\frac{m + m'}{m'}\right)^{\frac{1}{2}} \left\{\left(\frac{2g\tau^2}{\pi^2} y' - y'^2\right)^{\frac{1}{2}} + \frac{g\tau^2}{\pi^2} \text{ vers}^{-1} \frac{\pi^2 y'}{g\tau^2}\right\}.$$

Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 51.

(7) A heavy particle P descends down a smooth inclined plane CA, (fig. 205), forming the upper surface of a body CAE which is capable of sliding freely down a smooth inclined plane OB, with which its lower flat surface is in contact; to determine the motion of the particle and of the body, both of

which are supposed to be initially at rest, and the pressure of the particle on the body.

Let C be the initial position of the particle on the body, and O the initial position of the point A of the body; let $\angle BAC = \alpha$; $\angle OBF = \beta$, BF being horizontal; $OA = s$, $CP = s'$, $R =$ the required pressure; $m =$ the mass of the particle, $m' =$ that of the body; $t =$ the time from the commencement of the motion. Then

$$s = \tfrac{1}{2} gt^2 \, \frac{(m + m') \sin \beta + m \cos \alpha \sin (\alpha - \beta)}{m \sin^2 \alpha + m'},$$

$$s' = \tfrac{1}{2} gt^2 \, \frac{(m + m') \sin \alpha \cos \beta}{m \sin^2 \alpha + m'},$$

$$R = \frac{mm'g \cos \alpha \cos \beta}{m \sin^2 \alpha + m'}.$$

Euler; *Ibid.* p. 35.

(8) Any number of heavy particles, $P, P', P'', \ldots$, (fig. 206), are descending down a smooth inclined plane BA, forming the upper surface of a body BAC capable of sliding freely along a smooth horizontal plane OE, with which its lower flat surface is in contact; to determine the motion of the particles and of the body, and the pressures which they exert upon the body.

Let $R, R', R'', \ldots$ be the pressures, and $m, m', m'', \ldots$ the masses of the particles $P, P', P'', \ldots$; let $OA = S$, $BP = s$, $BP' = s'$, $BP'' = s''$, ..., O being a fixed point in the line OE, and B on the inclined plane BA; $\angle BAC = \alpha$; $M =$ the mass of the body. Then

$$S = \tfrac{1}{2} gt^2 \, \frac{(m + m' + m'' + \ldots) \sin \alpha \cos \alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2 \alpha} + At + B,$$

$$s = \tfrac{1}{2} gt^2 \, \frac{(M + m + m' + m'' + \ldots\ldots) \sin \alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2 \alpha} + at + b,$$

$$s' = \tfrac{1}{2} gt^2 \, \frac{(M + m + m' + m'' + \ldots\ldots) \sin \alpha}{M + (m + m' + m'' + \ldots\ldots) \sin^2 \alpha} + a't + b',$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{R}{m} = \frac{R'}{m'} = \frac{R''}{m''} = \ldots\ldots = \frac{Mg \cos \alpha}{M + (m + m' + m'' + \ldots) \sin^2 \alpha}.$$

The quantities A, B, a, b, a', b', ... are arbitrary constants, to be determined from the initial circumstances of the motion.

Euler; *Ibid.* p. 40.

(9) If a chain of considerable length be suspended from the top of a tower and then let fall, to find the velocity at any time.

Let x = the length and v = the velocity of the portion of the chain which is at any time in motion, a = the initial value of x, and r = the radius of the earth: then

$$v^2 = 2gr \log\left(\frac{a+r}{x+r}\right).$$

(10) A thin hollow ring, the plane of which is vertical, and which contains a bead, is placed upon a smooth horizontal plane: to find the period of the bead's oscillation, supposing it to have been placed initially near the lowest point of the ring.

If a be the radius of the ring, μ its mass, and m the mass of the bead, it will oscillate isochronously with a perfect pendulum the length of which is equal to

$$\frac{\mu a}{m+\mu}.$$

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

(11) A circular board lies on a smooth table: in the board is carved a circular concentric groove of radius a; a massive molecule is projected along this groove with a velocity v; to determine the horizontal pressure between the board and molecule.

The required pressure is constant and equal to

$$\tfrac{1}{2} M \frac{v^2}{a},$$

where M is the harmonic mean between the masses of the board and the molecule.

CHAPTER IX.

MOTION OF RIGID BODIES.

SECT. 1. *Single Body.*

(1) A CYLINDER descends down a perfectly rough inclined plane by the action of gravity, its axis being horizontal; to determine the motion of the cylinder and the friction at any time of its descent.

Let G (fig. 207) be the centre of gravity of the cylinder at any instant of its descent; OA the course of the point of contact H of the circular section of the cylinder through G down the inclined plane; let $OH = x$, $\alpha =$ the angle of inclination of OA to the horizon, $\theta =$ the whole angle through which the cylinder has revolved about its centre of gravity in moving from O to H; $a =$ the radius of the cylinder, $k =$ its radius of gyration about its axis. Let F denote the friction of the plane on the cylinder, which, from the signification of perfect roughness, is supposed to be sufficient to prevent sliding, and m the mass of the cylinder.

Then for the motion of the cylinder we have, resolving forces parallel to OA,

$$m\frac{d^2x}{dt^2} = mg \sin \alpha - F \dots\dots\dots\dots\dots\dots\dots(1);$$

and, taking moments about G,

$$mk^2\frac{d^2\theta}{dt^2} = Fa \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

But, since F is sufficiently great to secure perfect rolling, we must evidently have $x = a\theta$; and therefore, by (2),

$$mk^2\frac{d^2x}{dt^2} = Fa^2;$$

hence from (1) we get

$$ma^2 \frac{d^2x}{dt^2} = ma^2g \sin \alpha - mk^2 \frac{d^2x}{dt^2},$$

$$(a^2 + k^2) \frac{d^2x}{dt^2} = a^2g \sin \alpha,$$

or, since $k^2 = \frac{1}{2}a^2$, $\qquad 3 \frac{d^2x}{dt^2} = 2g \sin \alpha;$

and therefore, integrating twice, and supposing $\frac{dx}{dt} = 0$ when H is at O, we have

$$x = \frac{1}{3}gt^2 \sin \alpha,$$

and therefore $\qquad \theta = \dfrac{gt^2 \sin \alpha}{3a} \ldots\ldots\ldots\ldots (3);$

hence we have also, from (2) and (3),

$$F = \frac{mk^2}{a} \frac{d^2\theta}{dt^2} = \frac{2mgk^2 \sin \alpha}{3a^2} = \frac{1}{3}mg \sin \alpha.$$

(2) A globe descends from instantaneous rest down the surface of a perfectly rough hemispherical bowl, the centre of the globe always remaining in the same vertical plane; to determine the velocity of the globe at any position of its descent.

Let ABA' (fig. 208) be the vertical section of the bowl made by the plane in which the centre C of the globe is always situated, O being the centre of the bowl and OA a horizontal radius. Let M be the point in which the globe touches the bowl at any time of its motion, B being the initial position of M. Draw the radii OB, OCM; and let $C'C$ be the circular arc described by the centre of gravity of the globe. Let $\angle AOM = \theta$, $\angle AOB = \alpha$, $C'C = s$, $a =$ the radius of the globe, $r =$ the radius of the bowl, $\phi =$ the angle which the globe has described about its centre of gravity in the motion from B to M, $m =$ the mass of the globe; $F =$ the friction of the bowl upon the globe at the point M, which is supposed to be sufficiently great to prevent all sliding.

Then for the motion of the centre of gravity of the globe, which will not be affected by our supposing all the impressed forces to be applied at C in their proper directions,

$$m\frac{d^2s}{dt^2} = -F + mg\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1);$$

and, for the motion of the globe about its centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Fa \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

From the points B and M, draw two indefinite straight lines BkB' and MkT, tangents to the section of the hemispherical bowl; along BB' measure a length Bm equal to the circular arc BM; then, if we were to conceive the globe to roll from B along the length Bm, and then Bm to be applied along BM so as to coincide with it, mB' being, as soon as m coincides with M, a tangent both to the circle AMA' and to the globe; it is evident that the globe would have revolved about its centre through the same angle as by its actual motion of rolling down the arc BM. Now by rolling along Bm it would have revolved about its centre through an angle $\frac{Bm}{a} = \frac{BM}{a} = \frac{r}{a}(\theta - \alpha)$; and, by the transference of m to M, it would have revolved through an angle equal to $\angle B'kM = \angle BOM = \theta - \alpha$, in an opposite direction. Hence we see that the whole actual angle through which the globe revolves about its centre in its actual motion from B to M, is equal to $\frac{r-a}{a}(\theta - \alpha) = \phi$.

Hence, putting for ϕ its value in (2), we have

$$\frac{mk^2}{a}(r-a)\frac{d^2\theta}{dt^2} = Fa \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

Again, it is clear from the geometry that $s = (r-a)(\theta - \alpha)$, and therefore, from (1),

$$m(r-a)\frac{d^2\theta}{dt^2} = -F + mg\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Eliminating F between (3) and (4), we obtain

$$m(r-a)\frac{d^2\theta}{dt^2} + \frac{mk^2}{a^2}(r-a)\frac{d^2\theta}{dt^2} = mg\cos\theta,$$

$$\left(1 + \frac{k^2}{a^2}\right)(r-a)\frac{d^2\theta}{dt^2} = g\cos\theta,$$

or, since k^2 is equal to $\frac{2}{5}a^2$,

$$\frac{d^2\theta}{dt^2} = \frac{5g\cos\theta}{7(r-a)} \qquad \text{(5)};$$

multiplying by $2\dfrac{d\theta}{dt}$, and integrating,

$$\frac{d\theta^2}{dt^2} = C + \frac{10g\sin\theta}{7(r-a)};$$

but, when $\theta = \alpha$, $\dfrac{d\theta}{dt}$ is equal to zero; hence

$$0 = C + \frac{10g\sin\alpha}{7(r-a)},$$

and therefore $$\frac{d\theta^2}{dt^2} = \frac{10g(\sin\theta - \sin\alpha)}{7(r-a)};$$

whence $$\frac{ds^2}{dt^2} = \frac{10g}{7}(r-a)(\sin\theta - \sin\alpha);$$

and therefore also, if $s' =$ the arc BM,

$$\frac{ds'^2}{dt^2} = r^2\frac{d\theta^2}{dt^2} = \frac{10r^2g(\sin\theta - \sin\alpha)}{7(r-a)}.$$

For the magnitude of the friction at any time, we have, from (3),

$$F = \frac{mk^2}{a^2}(r-a)\frac{d^2\theta}{dt^2}$$

$$= \frac{5mgk^2\cos\theta}{7a^2}, \text{ by the equation (5).}$$

(3) A heterogeneous sphere rolls along a perfectly rough horizontal plane, its rotatory motion taking place always about an instantaneous axis normal to the vertical plane which passes through its geometrical centre and its centre of gravity; to determine its angular velocity for any position in its path.

Let C (fig. 209) be the geometrical centre and G the centre of gravity of the sphere at any time; S the point of contact of the vertical section of the sphere containing C and G with the horizontal plane; OSE the rectilinear locus of the points of contact; CGA a radius of the sphere; GM, CS, perpendiculars upon the plane.

Let F denote the friction of the plane at any time upon the sphere, estimated in the direction EO, and R the vertical reaction of the plane; let m = the mass of the sphere; k = the radius of gyration about an axis through G at right angles to the vertical section containing C and G; $OM=x$, $GM=y$, $CS=CA=a$, $\angle AGM = \angle ACS = \phi$, $CG=c$.

Then for the motion of the sphere we have, resolving forces parallel to OE,

$$m\frac{d^2x}{dt^2} = -F \quad\ldots\ldots\ldots\ldots (1);$$

resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg \quad\ldots\ldots\ldots\ldots (2);$$

and, taking moments about the centre of gravity,

$$mk^2\frac{d^2\phi}{dt^2} = Fy - Rc\sin\phi \quad\ldots\ldots\ldots\ldots (3).$$

Now, since the friction is supposed to be sufficiently rough to prevent all sliding, we have from the geometry,

$$x + c\sin\phi = b + a\phi,$$

b being the value of x when $\phi=0$; and therefore

$$\frac{dx}{dt} = a\frac{d\phi}{dt} - c\cos\phi\frac{d\phi}{dt},$$

$$\frac{d^2x}{dt^2} = (a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}.$$

Again, from the geometry,

$$y = a - c\cos\phi,$$

and therefore

$$\frac{dy}{dt} = c\sin\phi\frac{d\phi}{dt}, \qquad \frac{d^2y}{dt^2} = c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2}.$$

Hence, from (1),

$$F = -m\left\{(a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}\right\},$$

and, from (2),

$$R = m\left(\frac{d^2y}{dt^2} + g\right) = m\left(c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2} + g\right).$$

Substituting these expressions for R and F in (3), we get

$$k^2 \frac{d^2\phi}{dt^2} = -\left\{(a - c\cos\phi)\frac{d^2\phi}{dt^2} + c\sin\phi\frac{d\phi^2}{dt^2}\right\}(a - c\cos\phi)$$

$$- c\sin\phi\left(c\sin\phi\frac{d^2\phi}{dt^2} + c\cos\phi\frac{d\phi^2}{dt^2} + g\right),$$

and, by simplification,

$$(a^2 + k^2 + c^2 - 2ac\cos\phi)\frac{d^2\phi}{dt^2} + ac\sin\phi\frac{d\phi^2}{dt^2} = -cg\sin\phi;$$

multiplying by $2\frac{d\phi}{dt}$, and integrating,

$$(a^2 + k^2 + c^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = C + 2cg\cos\phi.$$

Let $t = 0$ when $\phi = 0$, and let ω be the initial angular velocity about G; then

$$(a^2 + k^2 + c^2 - 2ac)\,\omega^2 = C + 2cg,$$

and therefore

$$(a^2 + c^2 + k^2 - 2ac\cos\phi)\frac{d\phi^2}{dt^2} = \{(a-c)^2 + k^2\}\,\omega^2 - 2cg\,(1 - \cos\phi),$$

which gives the angular velocity about G at any time in terms of the whole angle described.

Euler; *Nova Acta Acad. Petrop.* 1783; p. 119.

(4) A pendulum of any figure is firmly attached to a solid circular cylinder as an axis; this axis is placed horizontally within a hollow circular horizontal cylinder of larger diameter, and of which the surface is perfectly rough; in the hollow cylinder there is a slit, through which the pendulum hangs; supposing the initial position of the pendulum to be very nearly a position of equilibrium, to find the length of an isochronous simple pendulum.

Let g (fig. 210) denote the position of the centre of gravity of the pendulum and its axis, regarded as one mass, at any instant of the small motions; let O, c, denote the centres of the circular sections of the hollow and the solid cylinders made by a vertical plane through g; let gp meet at right angles the vertical line OAp, which cuts in A the circular section MAN of the hollow cylinder; join Oc and produce the line to a, which will be the

point of contact of the sections of the solid and the hollow cylinders; let e be the point of contact of the section of the solid cylinder in its lowest position with that of the hollow one, so that the arc Aa will be equal to the arc ea. Let $Op = x$, $gp = y$, $ce = ca = b$, $AO = Oa = a$; $\phi = \angle cfO$ made by producing cg to meet Op produced; $\angle AOa = \theta$, $cg = c$, $\angle ecf = \beta$; $m =$ the mass of the pendulum and its axis together, $k =$ the radius of gyration about g of the pendulum and axis regarded as one mass, $R =$ the pressure of the hollow upon the solid cylinder, $F =$ the friction of the hollow cylinder upon the solid one in the direction of a tangent to the arc aN.

Then for the motion of the system we have, resolving forces vertically,

$$m\frac{d^2x}{dt^2} = mg - R\cos\theta - F\sin\theta;$$

resolving forces horizontally,

$$m\frac{d^2y}{dt^2} = -R\sin\theta + F\cos\theta,$$

and, taking moments about g,

$$mk^2\frac{d^2\phi}{dt^2} = -Rc\sin(\theta+\phi) + F\{c\cos(\theta+\phi) - b\}.$$

Now, θ and ϕ being by the hypothesis small angles, we may neglect their second and higher powers in the equations, and we get

$$m\frac{d^2x}{dt^2} = mg - R - F\theta,$$

$$m\frac{d^2y}{dt^2} = -R\theta + F,$$

$$mk^2\frac{d^2\phi}{dt^2} = -Rc(\theta+\phi) + F(c-b).$$

Eliminating R and F between these three equations, we shall finally obtain, as far as the first order of small quantities,

$$k^2\frac{d^2\phi}{dt^2} + cg(\theta+\phi) = \left(\frac{d^2y}{dt^2} + g\theta\right)(c-b) \;\ldots\ldots\ldots\; (1).$$

But, from the geometry, it is clear that

$$y=(a-b)\sin\theta-c\sin\phi=(a-b)\,\theta-c\phi \qquad \text{nearly,}$$

and therefore, putting for brevity $a-b=e$,

$$\frac{d^2y}{dt^2}=e\frac{d^2\theta}{dt^2}-c\frac{d^2\phi}{dt^2};$$

hence the equation (1) becomes

$$(k^2+c^2-cb)\frac{d^2\phi}{dt^2}-e(c-b)\frac{d^2\theta}{dt^2}+cg\phi+bg\theta=0 \ldots\ldots\ldots (2).$$

Now, since there is no sliding, we may shew, by precisely the same method as in the case of problem (2), that

$$\phi+\beta=\frac{a-b}{b}\,\theta=\frac{e}{b}\,\theta,$$

$\phi+\beta$ being evidently the whole angle described by the solid cylinder about its axis in rolling from a to A. Hence, from (2), we have

$$\{k^2+(c-b)^2\}\frac{d^2\phi}{dt^2}+\frac{g(b^2+ce)}{e}\phi+\frac{b^2\beta g}{e}=0:$$

let
$$\frac{g\,(b^2+ce)}{e}\phi+\frac{b^2\beta g}{e}=\frac{g\,(b^2+ce)}{e}\psi;$$

then $\dfrac{d^2\phi}{dt^2}=\dfrac{d^2\psi}{dt^2}$, and the equation becomes

$$\{k^2+(c-b)^2\}\frac{d^2\psi}{dt^2}+\frac{g\,(b^2+ce)}{e}\psi=0,$$

or
$$\frac{e\,\{k^2+(c-b)^2\}}{b^2+ce}\frac{d^2\psi}{dt^2}+g\psi=0.$$

Hence, if l denote the length of a perfect pendulum isochronous with the period of ϕ, and therefore of ψ, we shall have

$$l=\frac{e\,\{k^2+(c-b)^2\}}{b^2+ce}=\frac{(a-b)\,\{k^2+(c-b)^2\}}{b^2+c\,(a-b)}.$$

Euler; *Acta Acad. Petrop.* 1780; P. 2; p. 164.

(5) At the extremities A and B of a uniform beam AB, (fig. 211), are two small rings, capable of sliding along the horizontal and vertical rods Ox, Oy; the friction between the ends

of the beam and the rods is equal to the normal pressure on each; to determine the motion of the beam.

Let G be the centre of gravity of the beam; draw GH vertical; let $OH = x$, $GH = y$, $AG = BG = a$, $\angle BAO = \theta$; let R, S, be the normal reactions of the rods Ox, Oy, and therefore R, S, the frictions along xO, Oy; let $m =$ the mass of the beam, $k =$ the radius of gyration about G.

Then, for the motion of the beam,

$$m\frac{d^2x}{dt^2} = S - R \qquad (1),$$

$$m\frac{d^2y}{dt^2} = S + R - mg \qquad (2),$$

$$mk^2\frac{d^2\theta}{dt^2} = (S - R)\, a\cos\theta + (S + R)\, a\sin\theta \qquad (3).$$

Substituting the values of $S - R$ and $S + R$ from (1) and (2) in the equation (3), we get

$$k^2\frac{d^2\theta}{dt^2} = a\cos\theta\frac{d^2x}{dt^2} + \left(g + \frac{d^2y}{dt^2}\right) a\sin\theta \qquad (4).$$

But $$x = a\cos\theta,\quad \frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt},$$

$$\frac{d^2x}{dt^2} = -a\cos\theta\frac{d\theta^2}{dt^2} - a\sin\theta\frac{d^2\theta}{dt^2};$$

and $$y = a\sin\theta,\quad \frac{dy}{dt} = a\cos\theta\frac{d\theta}{dt},$$

$$\frac{d^2y}{dt^2} = -a\sin\theta\frac{d\theta^2}{dt^2} + a\cos\theta\frac{d^2\theta}{dt^2}.$$

Substituting these values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, in the equation (4), we have

$$k^2\frac{d^2\theta}{dt^2} + a^2\frac{d\theta^2}{dt^2} = ag\sin\theta;$$

or, changing the independent variable from t to θ,

$$-k^2\frac{\dfrac{d^2t}{d\theta^2}}{\dfrac{dt^3}{d\theta^3}} + \frac{a^2}{\dfrac{dt^2}{d\theta^2}} = ag\sin\theta:$$

multiplying both sides of the equation by $2\epsilon^{\frac{2a^2}{k^2}\theta}$, we obtain

$$\frac{d}{d\theta}\left\{\frac{k^2\epsilon^{\frac{2a^2}{k^2}\theta}}{\frac{dt^2}{d\theta^2}}\right\} = 2ag\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta;$$

integrating,

$$\frac{k^2\epsilon^{\frac{2a^2}{k^2}\theta}}{\frac{dt^2}{d\theta^2}} = C + 2ag\int\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta\,d\theta \quad\ldots\ldots\ldots\ldots\ (5).$$

Now, integrating by parts, we shall get

$$\int\epsilon^{\frac{2a^2}{k^2}\theta}\sin\theta\,d\theta = \frac{k^4}{k^4+4a^4}\left\{\frac{2a^2}{k^2}\sin\theta - \cos\theta\right\}\epsilon^{\frac{2a^2}{k^2}\theta};$$

hence, from (5),

$$k^2\epsilon^{\frac{2a^2}{k^2}\theta}\frac{d\theta^2}{dt^2} = \frac{2agk^4}{k^4+4a^4}\left\{C' + \left(\frac{2a^2}{k^2}\sin\theta - \cos\theta\right)\epsilon^{\frac{2a^2}{k^2}\theta}\right\},$$

$$\epsilon^{\frac{2a^2}{k^2}\theta}\frac{d\theta^2}{dt^2} = \frac{2agk^2}{k^4+4a^4}\left\{C' + \left(\frac{2a^2}{k^2}\sin\theta - \cos\theta\right)\epsilon^{\frac{2a^2}{k^2}\theta}\right\}.$$

Since $a^2 = 3k^2$, we have

$$\epsilon^{6\theta}\frac{d\theta^2}{dt^2} = \frac{6g}{37a}\{C' + (6\sin\theta - \cos\theta)\epsilon^{6\theta}\}.$$

Suppose that α, ω, are simultaneous values of θ, $\frac{d\theta}{dt}$; then

$$\epsilon^{6\alpha}\omega^2 = \frac{6g}{37a}\{C' + (6\sin\alpha - \cos\alpha)\epsilon^{6\alpha}\};$$

hence

$$\epsilon^{6\theta}\frac{d\theta^2}{dt^2} - \epsilon^{6\alpha}\omega^2 = \frac{6g}{37a}\{(6\sin\theta - \cos\theta)\epsilon^{6\theta} - (6\sin\alpha - \cos\alpha)\epsilon^{6\alpha}\},$$

which determines the angular velocity of the beam at every position in its descent.

(6) To determine the motion of a cylinder upon a perfectly rough plane of indefinite extent, which, having been initially horizontal, has a uniform motion downwards about a fixed horizontal axis within itself, to which the line of contact of the

cylinder and plane is always parallel, and with which it initially coincides.

Let a vertical plane at right angles to the axis of revolution, and passing through the centre of gravity C of the cylinder, cut the revolving plane at any instant of the motion in the line OA, (fig. 212); let Ox be the initial position of OA; draw CM, CN, at right angles to Ox, OA; let $OM = x$, $CM = y$, $ON = r$, $CN = a$, $\omega =$ the angular velocity of OA about O, $\angle AOx = \omega t$, $k =$ the radius of gyration of the cylinder about its axis, $R =$ the normal reaction of the plane in the direction NC, $T =$ the tangential reaction along NO, $\theta =$ the angle through which the cylinder has revolved about its centre of gravity at the end of the time t, $m =$ the mass of the cylinder.

Then, for the motion of the cylinder, we have

$$m\frac{d^2x}{dt^2} = R \sin \omega t - T \cos \omega t \quad\text{.....................(1)},$$

$$m\frac{d^2y}{dt^2} = mg - R \cos \omega t - T \sin \omega t \quad\text{...............(2)},$$

$$mk^2 \frac{d^2\theta}{dt^2} = Ta \quad\text{.......................................(3)}.$$

Again, the angle through which the cylinder would revolve about C, by rolling along Ox through a space r, would be $\frac{r}{a}$, and that due to the motion of Ox into the position OA would be ωt; hence $\frac{r}{a} + \omega t$ is the angle through which it has actually revolved at the end of the time t; or

$$\theta = \frac{r}{a} + \omega t \quad\text{..............................(4)}.$$

From the equations (1) and (2), we have

$$\sin \omega t \frac{d^2x}{dt^2} - \cos \omega t \frac{d^2y}{dt^2} = \frac{R}{m} - g \cos \omega t \quad\text{............ (5)};$$

and, from (1), (2), (3),

$$\cos \omega t \frac{d^2x}{dt^2} + \sin \omega t \frac{d^2y}{dt^2} = g \sin \omega t - \frac{k^2}{a} \frac{d^2\theta}{dt^2};$$

but, from (4), we get $$a\frac{d^2\theta}{dt^2}=\frac{d^2r}{dt^2};$$

hence $$\cos\omega t\frac{d^2x}{dt^2}+\sin\omega t\frac{d^2y}{dt^2}=g\sin\omega t-\frac{k^2}{a^2}\frac{d^2r}{dt^2}\quad\ldots\ldots\ldots(6).$$

From the geometry it is clear that

$$x=r\cos\omega t+a\sin\omega t,\qquad y=r\sin\omega t-a\cos\omega t;$$

differentiating these expressions twice with respect to t, we shall get

$$\frac{d^2x}{dt^2}=\cos\omega t\frac{d^2r}{dt^2}-2\omega\sin\omega t\frac{dr}{dt}-\omega^2r\cos\omega t-a\omega^2\sin\omega t,$$

$$\frac{d^2y}{dt^2}=\sin\omega t\frac{d^2r}{dt^2}+2\omega\cos\omega t\frac{dr}{dt}-\omega^2r\sin\omega t+a\omega^2\cos\omega t:$$

substituting these expressions for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, in the equations (5) and (6), we obtain

$$2\omega\frac{dr}{dt}+a\omega^2=g\cos\omega t-\frac{R}{m}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and $$\frac{a^2+k^2}{a^2}\frac{d^2r}{dt^2}-\omega^2r=g\sin\omega t\quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Since $a^2=2k^2$, the equation (8) becomes

$$\frac{d^2r}{dt^2}-\tfrac{2}{3}\omega^2r=\tfrac{2}{3}g\sin\omega t;$$

the integral of this equation is

$$r=-\frac{2g}{5\omega^2}\sin\omega t+Ce^{\omega' t}+C'e^{-\omega' t},$$

where $\omega'=(\frac{2}{3})^{\frac{1}{2}}\omega$, and C, C', are arbitrary constants. If we determine C and C' from the conditions that $r=0$, $\frac{dr}{dt}=0$, initially, we shall have

$$r=-\frac{2g}{5\omega^2}\sin\omega t+\frac{3^{\frac{1}{2}}g}{5\omega^2 2^{\frac{1}{2}}}(e^{\omega' t}-e^{-\omega' t}),$$

which determines the position of the cylinder at any time before it detaches itself from the revolving plane: differentiating with respect to t,

$$\frac{dr}{dt} = -\frac{2g}{5\omega}\cos\omega t + \frac{g}{5\omega}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right);$$

hence, from (7), we obtain

$$\frac{R}{m} = g\cos\omega t - a\omega^2 + \tfrac{4}{5}g\cos\omega t - \frac{2g}{5}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right)$$

$$= \tfrac{9}{5}g\cos\omega t - a\omega^2 - \frac{2g}{5}\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right),$$

which gives the value of R for any time of the motion of the cylinder upon the plane: when $R = 0$, or when the cylinder leaves the plane,

$$9g\cos\omega t = 5a\omega^2 + 2g\left(\epsilon^{\omega' t} + \epsilon^{-\omega' t}\right),$$

an equation which fixes the epoch of the separation.

(7) A sphere is projected directly down an inclined plane with a motion both of translation and of rotation; the motion of rotation is the same in point of direction as that which would correspond to perfect downward rolling, but greater in magnitude: to determine the motion of the sphere, having given the coefficients both of statical and of dynamical friction between the sphere and the inclined plane.

Let OA (fig. 213) be the inclined plane, C the position of the sphere's centre, and M its point of contact with OA at the end of a time t from the beginning of the motion. Let $\mu =$ the coefficient of dynamical friction between the sphere and the plane, $a =$ the radius of the sphere, $OM = s$, $\phi =$ the angle through which the sphere has revolved at the end of the time t about its centre of gravity, $R =$ the normal reaction of the plane, $m =$ the mass of the sphere, $\alpha =$ the inclination of the plane to the horizon.

Then, for the motion of the sphere, we have

$$m\frac{d^2 s}{dt^2} = mg\sin\alpha + \mu R \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$mk^2\frac{d^2\phi}{dt^2} = -\mu a R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2):$$

but, since C has no motion at right angles to the plane, we have also $R = mg \cos \alpha$; hence, from (1),

$$\frac{d^2 s}{dt^2} = g (\sin \alpha + \mu \cos \alpha) \quad \text{................(3)},$$

and, from (2).

$$k^2 \frac{d^2 \phi}{dt^2} = - \mu a g \cos \alpha \quad \text{....................(4)}.$$

Integrating the equation (3) with respect to t, and denoting the initial value of $\frac{ds}{dt}$ by c, we get

$$\frac{ds}{dt} = gt (\sin \alpha + \mu \cos \alpha) + c \quad \text{................(5)};$$

similarly from (4), ω denoting the initial value of $\frac{d\phi}{dt}$,

$$\frac{d\phi}{dt} = \omega - \frac{\mu a g}{k^2} t \cos \alpha \quad \text{....................(6)}.$$

As soon as, by the increase of t, $\frac{ds}{dt}$ becomes equal to $a\frac{d\phi}{dt}$, the motion will change its character, and our present equations will cease to be applicable. This event will take place when

$$gt (\sin \alpha + \mu \cos \alpha) + c = a\omega - \frac{\mu a^2 g t}{k^2} \cos \alpha,$$

or $$t = \frac{k^2 (a\omega - c)}{\mu g (a^2 + k^2) \cos \alpha + k^2 g \sin \alpha} = t' \text{ suppose.}$$

For all values of t not greater than t', we have from (5) and (6), s and ϕ being considered to be initially zero,

$$s = \tfrac{1}{2} g t^2 (\sin \alpha + \mu \cos \alpha) + ct,$$

$$\phi = \omega t - \frac{\mu a g t^2}{2k^2} \cos \alpha;$$

the values of s and ϕ at the end of the first period of the motion will be obtained from these expressions by substituting t' in place of t.

When $\frac{ds}{dt}$ becomes equal to $a\frac{d\phi}{dt}$, there evidently exists at

that instant no sliding between the plane and the sphere; and therefore, before dynamical friction can again come into play, the statical friction between the sphere and the plane must be overcome. First, let us suppose that the statical friction is sufficiently great to secure perfect rolling; and let F denote the tangential reaction of the plane against the descent of the sphere. The equations of motion will be,

$$m\frac{d^2s}{dt^2} = mg\sin\alpha - F \dots\dots\dots\dots\dots\dots\dots (1),$$

$$mk^2\frac{d^2\phi}{dt^2} = aF \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2).$$

Also, there being no sliding, it is clear that

$$a\frac{d\phi}{dt} = \frac{ds}{dt}, \quad a\frac{d^2\phi}{dt^2} = \frac{d^2s}{dt^2};$$

hence from (2) there is

$$mk^2\frac{d^2s}{dt^2} = a^2F,$$

and therefore, from (1),

$$(a^2 + k^2)\frac{d^2s}{dt^2} = a^2g\sin\alpha \dots\dots\dots\dots\dots\dots\dots (3);$$

and therefore, also,

$$(a^2 + k^2)\frac{d^2\phi}{dt^2} = ag\sin\alpha \dots\dots\dots\dots\dots\dots\dots (4).$$

Integrating, we get from (3) and (4),

$$\frac{ds}{dt} = \frac{a^2g\sin\alpha}{a^2+k^2}t + c',$$

$$\frac{d\phi}{dt} = \frac{ag\sin\alpha}{a^2+k^2}t + \omega';$$

where c', ω', are the values of $\frac{ds}{dt}$, $\frac{d\phi}{dt}$, at the end of the first stage of the motion, the time being now reckoned from the commencement of the second period: c' is clearly equal to $a\omega'$.

Integrating again, we get

$$s = \frac{\frac{1}{2}a^2gt^2 \sin \alpha}{a^2 + k^2} + c't + s';$$

$$\phi = \frac{\frac{1}{2}agt^2 \sin \alpha}{a^2 + k^2} + \omega' t + \phi',$$

s', ϕ', being the values of ϕ, s, at the end of the first period.

Also $$F = \frac{mk^2}{a}\frac{d^2\phi}{dt^2} = \frac{mk^2 g \sin \alpha}{a^2 + k^2},$$

which is the value of the statical friction necessary to secure perfect rolling in the second stage of the motion.

If the statical friction be less than this, dynamical friction will arise, and will evidently exert itself *up* the plane. Hence, for the motion,

$$\frac{d^2s}{dt^2} = g(\sin \alpha - \mu \cos \alpha) \text{ .... (A)},$$

$$k^2 \frac{d^2\phi}{dt^2} = \mu a g \cos \alpha \text{ .... (B)}.$$

It may be easily ascertained that the coefficient of g in the expression for $\frac{d^2s}{dt^2}$ is positive; for the coefficient of friction necessary for perfect rolling

$$= \frac{F}{R} = \frac{F}{mg \cos \alpha} = \frac{k^2 \tan \alpha}{a^2 + k^2},$$

and, since μ is less than this by hypothesis, we have

$$\mu \angle \frac{k^2}{a^2 + k^2} \tan \alpha, \text{ and therefore } \mu \angle \tan \alpha,\ \mu \cos \alpha \angle \sin \alpha.$$

From (A) and (B) we have

$$\frac{ds}{dt} = gt(\sin \alpha - \mu \cos \alpha) + c',$$

$$\frac{d\phi}{dt} = \frac{\mu a g t}{k^2} \cos \alpha + \omega'.$$

It may be readily seen that $\frac{ds}{dt} - a\frac{d\phi}{dt}$ never becomes zero in

the second stage of the motion, but is always positive; for, bearing in mind that $c' = a\omega'$,

$$\frac{ds}{dt} - a\,\frac{d\phi}{dt} = gt\left(\sin\alpha - \mu\cos\alpha - \mu\,\frac{a^2}{k^2}\cos\alpha\right)$$

$$= gt\cos\alpha\,\frac{a^2+k^2}{k^2}\left(\frac{k^2}{a^2+k^2}\tan\alpha - \mu\right);$$

hence the sphere will always rotate too slowly, in comparison with the velocity of translation, to correspond to perfect rolling.

If s' denote the space through which the sphere descends along the plane in the second stage of the motion, in consequence of sliding,

$$\frac{ds'}{dt} = \frac{ds}{dt} - a\,\frac{d\phi}{dt} = gt\left(\sin\alpha - \mu\cos\alpha\,\frac{a^2+k^2}{k^2}\right),$$

$$s' = \tfrac{1}{2}gt^2\left(\sin\alpha - \mu\cos\alpha\,\frac{a^2+k^2}{k^2}\right).$$

Euler; *Acta Acad. Petrop.* P. II. p. 131; 1781.

(8) A sphere, revolving about a horizontal axis, is placed upon an imperfectly rough plane inclined to the horizon at an angle $\tan^{-1}\mu$, where μ is the coefficient of dynamical friction, the direction of the sphere's rotation being opposite to that which would correspond to perfect downward rolling; to determine the motion of the sphere.

The notation remaining the same as in the preceding problem, we shall have, for the motion of the sphere,

$$m\,\frac{d^2s}{dt^2} = mg\sin\alpha - \mu R,$$

$$mk^2\,\frac{d^2\phi}{dt^2} = -\mu a R;$$

but $R = mg\cos\alpha$; hence we have

$$\frac{d^2s}{dt^2} = g\,(\sin\alpha - \mu\cos\alpha) = 0, \text{ by hypothesis;}$$

and $$\frac{d^2\phi}{dt^2} = -\frac{\mu a g\cos\alpha}{k^2}.$$

Integrating with respect to t, we get

$$\frac{ds}{dt} = C, \quad \frac{d\phi}{dt} = C' - \frac{\mu agt}{k^2} \cos \alpha,$$

where C, C', are arbitrary constants; but, initially, $\frac{ds}{dt} = 0$, $\frac{d\phi}{dt} = \omega$; and therefore $C = 0$, $C' = \omega$; hence

$$\frac{ds}{dt} = 0, \quad \frac{d\phi}{dt} = \omega - \frac{\mu agt}{k^2} \cos \alpha.$$

In the preceding investigation we have supposed the friction of the plane upon the sphere to act upwards; this will cease to be the case when $\frac{ds}{dt} + a\frac{d\phi}{dt}$, or, since $\frac{ds}{dt} = 0$, when $\frac{d\phi}{dt}$ becomes zero. Hence we see that, for a time equal to

$$\frac{k^2\omega}{\mu ag \cos \alpha},$$

the centre of the sphere will remain stationary, and that at the end of this time the angular velocity of the sphere will become zero.

Before proceeding to investigate the nature of the motion after the end of the stationary period of the centre of the sphere, it will be necessary to determine the amount of upward friction requisite to cause the sphere subsequently to assume a motion of perfect rolling. The coefficient of friction requisite for this purpose, (see the preceding problem,) is equal to

$$\frac{k^2 \tan \alpha}{a^2 + k^2}.$$

But μ is equal to $\tan \alpha$, and therefore exceeds the requisite magnitude. The sphere will then proceed subsequently to roll down the plane without sliding; and the space described by its centre, at the end of a time t from the termination of its stationary interval, will be equal to

$$\frac{\frac{1}{2}a^2gt^2 \sin \alpha}{a^2 + k^2},$$

which, since $k^2 = \frac{2}{5}a^2$, is equal to $\frac{5}{14}gt^2 \sin \alpha$. Also, putting for k^2 its value, the stationary interval will have for its value $\dfrac{2a\omega}{5g \sin \alpha}$.

Euler; *Acta Acad. Petrop.* P. II. p. 131; 1781.

(9) A homogeneous sphere attracted towards a given centre of force varying directly as the distance, is projected with a given velocity along a plane passing through that centre, friction being such as to prevent all sliding; to determine the path described by the sphere.

Let O, (fig. 214), the centre of force, be taken as the origin of co-ordinates, and let Ox, Oy, which are at right angles to each other and in the plane along which the sphere rolls, be taken as the co-ordinate axes. Let C be the centre of the sphere at any time of its motion, P its point of contact with the plane xOy; join CO, CP, and draw PM parallel to yO; draw also Oz at right angles to the plane xOy. Let $OC = r$, $CP = c$, $OM = x$, $PM = y$, $m =$ the mass of the sphere, $\mu =$ the attraction of the central force upon a unit of mass collected at a point at a unit of distance; let X, Y, denote the friction of the plane on the sphere, estimated parallel to Ox, Oy; let ω', ω'', be the angular velocities of the sphere at any time about diameters parallel to Ox, Oy, estimated in the directions indicated by the arrows in the planes yOz, zOx; let $k =$ the radius of gyration of the sphere about a diameter.

It may be readily ascertained that the attraction on the whole sphere will be equal to a force μmr in the direction CO, and the resolved parts of this force parallel to Ox, Oy, are evidently $-\mu mx$, $-\mu my$. Hence, for the motion of the sphere, we have

$$m \frac{d^2x}{dt^2} = X - \mu mx \quad\quad (1),$$

$$m \frac{d^2y}{dt^2} = Y - \mu my \quad\quad (2),$$

$$mk^2 \frac{d\omega'}{dt} = cY \quad\quad (3),$$

$$mk^2 \frac{d\omega''}{dt} = -cX \quad\quad (4).[1]$$

[1] If ω', ω'', ω''', denote the angular velocities of a rigid body about three straight lines through its centre of inertia, parallel to the axes of x, y, z, respectively, and

But, since the friction is sufficiently great to prevent all sliding, it is clear that

$$\frac{dx}{dt} = c\omega'', \qquad \frac{dy}{dt} = -c\omega';$$

hence, from (3) and (4), we get

$$Y = -\frac{mk^2}{c^2}\frac{d^2y}{dt^2}, \qquad X = -\frac{mk^2}{c^2}\frac{d^2x}{dt^2},$$

and therefore, from (1), (2),

$$\frac{c^2+k^2}{c^2}\frac{d^2x}{dt^2} = -\mu x,$$

$$\frac{c^2+k^2}{c^2}\frac{d^2y}{dt^2} = -\mu y;$$

or, since $k^2 = \frac{2}{5}c^2$,

$$\frac{d^2x}{dt^2} = -\frac{5\mu}{7}x, \qquad \frac{d^2y}{dt^2} = -\frac{5\mu}{7}y:$$

the integrals of these equations are

$$x = A\sin(\lambda t + \epsilon),$$
$$y = A'\sin(\lambda t + \epsilon'),$$

δm denote an element of the mass, then (See Pratt's *Mechanical Philosophy*, First Edit. p. 428),

$$(\omega'''^2 - \omega''^2)\,\Sigma(\delta m\,.\,yz) + \left(\omega'''\omega' - \frac{d\omega''}{dt}\right)\Sigma(\delta m\,.\,xy)$$
$$-\left(\omega''\omega' + \frac{d\omega'''}{dt}\right)\Sigma(\delta m\,.\,xz) + \omega'''\omega''\,\Sigma(\delta m\,.\,y^2)$$
$$-\omega''\omega'''\,\Sigma(\delta m\,.\,z^2) + \frac{d\omega'}{dt}\Sigma\{\delta m\,.\,(y^2+z^2)\} = L,$$

where L denotes the moment of the forces about the first of the three straight lines.

Similar equations, *mutatis mutandis*, will hold in relation to the axes of y and z. If the body be a homogeneous sphere, as in the present problem,

$$\Sigma(\delta m\,.\,yz) = 0, \quad \Sigma(\delta m\,.\,xy) = 0, \quad \Sigma(\delta m\,.\,xz) = 0, \quad \Sigma(\delta m\,.\,y^2) - \Sigma(\delta m\,.\,z^2) = 0,$$

and therefore, adopting the notation of the text,

$$mk^2\frac{d\omega'}{dt} = L,$$

L being equal to cY. Similarly

$$mk^2\frac{d\omega''}{dt} = -cX.$$

A, A', ϵ, ϵ', being arbitrary constants, and λ denoting $\left(\frac{5\mu}{7}\right)^{\frac{1}{2}}$.

Let a, b, be the initial values of x, y, and α, β, of $\frac{dx}{dt}$, $\frac{dy}{dt}$; then

$$x = a \cos(\lambda t) + \frac{\alpha}{\lambda} \sin(\lambda t),$$

$$y = b \cos(\lambda t) + \frac{\beta}{\lambda} \sin(\lambda t).$$

From the last two equations we have,

$$\beta x - \alpha y = (a\beta - b\alpha) \cos(\lambda t),$$

and $$\lambda (bx - ay) = -(a\beta - b\alpha) \sin(\lambda t).$$

Squaring the last two equations and adding, we obtain, for the equation to the path of the sphere,

$$(\beta x - \alpha y)^2 + \lambda^2 (bx - ay)^2 = (a\beta - b\alpha)^2.$$

Thus we see that the sphere will describe an ellipse the centre of which coincides with the origin of co-ordinates.

(10) A wheel, of given radius a, the centre of gravity of which is at a distance c from its centre, rolls on a perfectly rough horizontal plane; to find the velocity of the centre, when the centre of gravity is vertically below it, in order that, when it is vertically above the centre, the normal pressure on the plane may be zero: to determine also the friction in these two positions of the centre of gravity, and the normal pressure when the centre of gravity is vertically below the centre.

Let k denote the radius of gyration about the centre of gravity, v the required velocity of the centre; F the friction, when the centre of gravity is lowest, and F', when it is highest; R the normal pressure when the centre of gravity is lowest: then

$$v^2 = \frac{a^2 g}{c} \cdot \frac{4c^2 + (a+c)^2 + k^2}{(a-c)^2 + k^2}, \quad F = 0, \quad F' = 0,$$

$$R = 2mg \cdot \frac{a^2 + 3c^2 + k^2}{(a-c)^2 + k^2}.$$

For a complete discussion of the Rolling motion of a cylinder on a rough plane and other interesting problems, the student is referred to a memoir by the Rev. Henry Moseley, in the *Philosophical Transactions of London;* Part 2, for 1851.

(11) A sphere is projected obliquely up a perfectly rough inclined plane: to find the equation to the path of the point of contact between the sphere and plane.

Let $\alpha =$ the inclination of the plane to the horizon, $V =$ the velocity of projection, $\beta =$ the inclination of the direction of projection to the horizontal line Ox in the inclined plane through the point O of projection; let Oy be drawn up the inclined plane at right angles to Ox.

Then the equation to the path of the point of contact will be

$$y = x \tan\beta - \frac{5}{14} \cdot \frac{gx^2 \sin\alpha}{V^2 \cos^2\beta}.$$

SECT. 2. *Several Bodies.*

(1) A cylinder rolls down a perfectly rough inclined plane, while a string coils round it which unwinds from an equal cylinder revolving about its axis which is fixed, the position of the latter cylinder being such that the string is parallel to the plane; to find the accelerative force of descent, the tension of the string, and the friction of the inclined plane.

Let O (fig. 215) be the centre of gravity of the descending cylinder at any time of its motion down the plane BA, M being its point of contact with the plane; let C be the centre of gravity of the other cylinder; join CO. Let $CO = x$ at any time t; $a =$ the radius of each of the cylinders, $\alpha =$ the inclination of the plane BA to the horizon, $T =$ the tension of the uncoiled string, $F =$ the friction of the inclined plane exerted upon the cylinder O at M in the direction MB; $m =$ the mass of each of the cylinders, and $k =$ the radius of gyration of each about its axis; let θ, θ', denote the angles through which the cylinders O, C, have revolved about their axes at the end of the time t.

Then, for the motion of the cylinder O, we have

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - F - T \quad\text{.....................(1)},$$

$$mk^2\frac{d^2\theta}{dt^2} = (F - T)\,a \quad\text{.............................(2)};$$

and, for the motion of the cylinder C,

$$mk^2\frac{d^2\theta'}{dt^2} = Ta \quad\text{..............................(3)}.$$

Multiplying the equations (1) and (3) by a and 2 respectively, and adding the resulting equations to (2), we get

$$k^2\frac{d^2\theta}{dt^2} + 2k^2\frac{d^2\theta'}{dt^2} + a\frac{d^2x}{dt^2} = ag\sin\alpha;$$

but, from the geometry, it is evident that

$$a\frac{d^2\theta}{dt^2} = \frac{d^2x}{dt^2}, \qquad a\frac{d^2\theta'}{dt^2} = 2\frac{d^2x}{dt^2};$$

hence we have $\qquad (a^2 + 5k^2)\dfrac{d^2x}{dt^2} = a^2g\sin\alpha,$

or, since $2k^2 = a^2$,

$$\frac{d^2x}{dt^2} = \tfrac{2}{7}g\sin\alpha.$$

Again, from (3),

$$T = \frac{mk^2}{a}\frac{d^2\theta'}{dt^2} = \frac{2mk^2}{a^2}\frac{d^2x}{dt^2} = m\frac{d^2x}{dt^2} = \tfrac{2}{7}mg\sin\alpha.$$

Lastly, from (2), $\quad F = T + \dfrac{mk^2}{a}\dfrac{d^2\theta}{dt^2} = T + \tfrac{1}{2}ma\dfrac{d^2\theta}{dt^2}$

$$= T + \tfrac{1}{2}m\frac{d^2x}{dt^2} = \tfrac{2}{7}mg\sin\alpha + \tfrac{1}{7}mg\sin\alpha$$

$$= \tfrac{3}{7}mg\sin\alpha.$$

(2) A string, the free end of which hangs through a ring and has a weight attached to it, is wound about a circular section of a cylinder, made by a vertical plane passing through the ring and through the centre of gravity of the cylinder; the cylinder rests upon a rough horizontal plane, and its diameter is equal to the

altitude of the ring above this plane: to determine the motion of the weight and of the cylinder, which are supposed to be initially in a state of instantaneous rest.

Let R (fig. 216) be the position of the ring; NRP the free portion of the string meeting in the point O the locus OMA of the point M in which the circular section of the cylinder touches the plane at any time; C the position at any time of the centre of gravity of the cylinder. Let a = the radius of the cylinder, $OP = y$, $OM = x$, ϕ = the angle through which the cylinder has revolved about its axis at the end of the time t, F = the action of the plane on the cylinder estimated in the direction AO, m = the mass of the cylinder, k = its radius of gyration about its axis, m' = the mass of P, T = the tension of the string.

Then, for the motion of the weight, we have

$$m'\frac{d^2y}{dt^2} = m'g - T \text{..........................} (1);$$

and for the motion of the cylinder, both in respect to translation and to rotation,

$$m\frac{d^2x}{dt^2} = -T - F \text{........................} (2),$$

$$mk^2\frac{d^2\phi}{dt^2} = Ta - Fa \text{........................} (3).$$

Now, since the horizontal plane along which the cylinder moves may be either perfectly or imperfectly rough, we shall have to consider two cases of the motion. We will first suppose the plane to be perfectly rough, that is, to be sufficiently rough to prevent all sliding.

Since the cylinder rolls without sliding, we must evidently have $dx = -a d\phi$; and therefore, by (3),

$$-mk^2\frac{d^2x}{dt^2} = Ta^2 - Fa^2 \text{..................} (4);$$

also, from (2),

$$-ma^2\frac{d^2x}{dt^2} = Ta^2 + Fa^2 \text{..................} (5),$$

and therefore, adding together these last two equations,

$$-m(a^2+k^2)\frac{d^2x}{dt^2}=2Ta^2 \dots\dots\dots\dots\dots\dots (6);$$

and therefore, by (1),

$$2m'a^2\frac{d^2y}{dt^2}-m(a^2+k^2)\frac{d^2x}{dt^2}=2m'a^2g:$$

but, if l denote the original length of the free string, it is clear that

$$x+y=l+a\phi,\quad \frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=a\frac{d^2\phi}{dt^2}=-\frac{d^2x}{dt^2},\quad \frac{d^2y}{dt^2}=-2\frac{d^2x}{dt^2};$$

hence we have

$$\{4m'a^2+m(a^2+k^2)\}\frac{d^2x}{dt^2}=-2m'a^2g \dots\dots\dots\dots\dots (7);$$

integrating, and bearing in mind that $\frac{dx}{dt}=0$ when $t=0$,

$$\{4m'a^2+m(a^2+k^2)\}\frac{dx}{dt}=-2m'a^2gt:$$

integrating again, and taking c for the initial value of x,

$$x=c-\frac{m'a^2gt^2}{4m'a^2+m(a^2+k^2)}.$$

We may easily get also

$$y=l-c+\frac{2m'a^2gt^2}{4m'a^2+m(a^2+k^2)};$$

and therefore, since $a\phi=x+y-l$,

$$\phi=\frac{m'agt^2}{4m'a^2+m(a^2+k^2)}.$$

Also, from (4) and (5),

$$2Fa^2=-m(a^2-k^2)\frac{d^2x}{dt^2},$$

and consequently, from (7),

$$F=\frac{mm'(a^2-k^2)g}{4m'a^2+m(a^2+k^2)}.$$

Also, from (6), we have

$$T=\frac{m(a^2+k^2)}{2a^2}\frac{d^2x}{dt^2}$$

$$=\frac{mm'g(a^2+k^2)}{4m'a^2+m(a^2+k^2)}.$$

We will now proceed to the consideration of the case when the friction of the plane is not sufficient to prevent sliding: and since the value which we obtained for F, the friction necessary to prevent sliding, is positive, therefore this force would act during the whole motion in the direction AO, which shews that, if the action of the plane on the cylinder be not sufficient to secure perfect rolling, the dynamical friction will be exerted in the direction AO. Let μ denote the coefficient of dynamical friction; then, putting μmg instead of F in the equations (2) and (3), we have

$$m\frac{d^2x}{dt^2} = -T - \mu mg,$$

$$mk^2\frac{d^2\phi}{dt^2} = Ta - \mu mag;$$

from these two equations, together with the equation (1), and the appropriate geometrical relations, we may easily shew that

$$\phi = \frac{1}{2}\frac{(1-2\mu)m' - \mu m}{mk^2 + m'(a^2+k^2)}agt^2,$$

$$x = c - \frac{1}{2}gt^2\frac{m'(2\mu a^2 + k^2) + m\mu k^2}{mk^2 + m'(a^2+k^2)},$$

$$y = l - c + \frac{1}{2}gt^2\frac{m'(a^2+k^2) - \mu m(a^2-k^2)}{mk^2 + m'(a^2+k^2)},$$

$$T = \frac{\mu a^2 + (1-\mu)k^2}{mk^2 + m'(a^2+k^2)}mm'g.$$

In a memoir by Fuss, from which this problem has been extracted, is discussed the more general case when the cylinder descends down an inclined plane, and the ring is replaced by a pulley of considerable inertia.

Fuss; *Nova Acta Acad. Petrop.* 1787; p. 176.

(3) A cylinder rolls, without sliding, down a moveable inclined plane, which rests on a perfectly smooth horizontal surface; to determine the motion of the plane and of the cylinder.

The axis of the cylinder is supposed to be horizontal, and a vertical plane, at right angles to the axis, to contain the centre

of gravity C (fig. 217) of the cylinder and the centre of gravity of the inclined plane: let Ox be the intersection of this vertical plane with the smooth surface which supports the inclined plane AB. Draw CM at right angles to Ox. Let R be the mutual normal action and reaction of the cylinder and inclined plane F the action of the plane on the cylinder along AB; let $m =$ the mass of the cylinder, $mk^2 =$ its moment of inertia about its axis; $m' =$ the mass of the inclined plane; $OM = x$, $CM = y$, $OA = x'$; $\theta =$ the angle through which the cylinder has revolved about its axis at the end of the time t; $\alpha =$ the inclination of AB to Ox, $a =$ the radius of the cylinder.

Then, for the motion of the cylinder, we have

$$m\frac{d^2x}{dt^2} = -R\sin\alpha + F\cos\alpha \text{ ....(1)},$$

$$m\frac{d^2y}{dt^2} = R\cos\alpha + F\sin\alpha - mg \text{ ....(2)},$$

$$mk^2\frac{d^2\theta}{dt^2} = Fa \text{ ....(3)};$$

and, for the motion of the inclined plane,

$$m'\frac{d^2x'}{dt^2} = R\sin\alpha - F\cos\alpha \text{ ....(4)}.$$

From the equations (1) and (4), we get

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0 \text{ ....(5)}.$$

Again, from the geometry, we see that

$$y\cos\alpha = a + (x - x')\sin\alpha;$$

hence
$$\cos\alpha\frac{d^2y}{dt^2} = \sin\alpha\left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right),$$

and therefore, by (5),

$$m'\cos\alpha\frac{d^2y}{dt^2} = (m + m')\sin\alpha\frac{d^2x}{dt^2} \text{ ....(6)}.$$

Also, since no sliding takes place between the cylinder and the plane, it is clear that

$$\frac{dx}{dt} = \frac{dx'}{dt} - a\cos\alpha\frac{d\theta}{dt},$$

and therefore $$a \cos \alpha \frac{d^2\theta}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2};$$

hence, by the aid of (5), we have

$$m'a \cos \alpha \frac{d^2\theta}{dt^2} = -(m+m') \frac{d^2x}{dt^2} \;\ldots\ldots\ldots\ldots(7).$$

Again, from (1) and (2),

$$m \cos \alpha \frac{d^2x}{dt^2} + m \sin \alpha \frac{d^2y}{dt^2} = F - mg \sin \alpha,$$

and therefore, by (3),

$$a \cos \alpha \frac{d^2x}{dt^2} + a \sin \alpha \frac{d^2y}{dt^2} = k^2 \frac{d^2\theta}{dt^2} - ag \sin \alpha:$$

substituting in this equation the expressions for $\frac{d^2y}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ given in (6) and (7), we obtain

$$\{m'a^2 \cos^2 \alpha + (m+m')(a^2 \sin^2 \alpha + k^2)\} \frac{d^2x}{dt^2} = -m'a^2 g \sin \alpha \cos \alpha:$$

which gives the value of $\frac{d^2x}{dt^2}$, which it appears therefore is constant during the whole motion; the values of $\frac{d^2x'}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, may now be readily obtained by the aid of the equations (5), (6), (7), and will be constant during the whole motion. Knowing the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, $\frac{d^2x'}{dt^2}$, we may immediately obtain the values of x, y, θ, x', in terms of t, if we have given the initial values of x, x', $\frac{dx}{dt}$, $\frac{dx'}{dt}$. The values of R and F may also be readily obtained from the equations (1), (2), (3), (4).

(4) A bullet is fired with a given velocity into a body in a direction passing through the centre of gravity of the body; the body is initially at rest and is capable of free motion, not being under the action of any forces; to determine the velocities of the bullet and of the body when the bullet has traversed any space within the body; the resistance of the body to the motion of the bullet being supposed to be a constant force.

Let k denote the constant retarding force, m the mass of the bullet, μ of the body, β the initial velocity of the bullet; then if u and v denote the velocities of the bullet and of the body when the bullet has traversed a space x within the body,

$$u = \frac{m\beta}{m+\mu} + \frac{\mu}{m+\mu}\left\{\beta^2 - \frac{2k}{m\mu}(m+\mu)\,x\right\}^{\frac{1}{2}},$$

$$v = \frac{m\beta}{m+\mu} - \frac{m}{m+\mu}\left\{\beta^2 - \frac{2k}{m\mu}(m+\mu)\,x\right\}^{\frac{1}{2}}.$$

Camus; *Mém. de l'Acad. des Sciences de Paris*, 1738, p. 147.

CHAPTER X.

DYNAMICAL PRINCIPLES.

SECT. 1. *Vis Viva.*

THE term Vis Viva was first introduced into the language of Mechanics by Leibnitz, in a memoir published in the *Acta Eruditorum* for the year 1695, entitled *Specimen dynamicum pro admirandis naturæ legibus circa corporum vires et mutuas actiones detegendis et ad suas causas revocandis*: it was intended by its author to signify the force of a body in actual motion, called otherwise its Vis Motrix or Moving Force, as distinguished from the statical pressure of a body, which has merely a tendency to motion, against a fixed obstacle; the statical force of a body he designated by the appellation of Vis Mortua. Leibnitz contended, in opposition to the received doctrine of the Cartesians, that the proper measure of the Vis Viva or Moving Force of a body, is the product of its mass into the square of its velocity, the measure adopted by the disciples of Descartes having been the same as that of the Quantity of Motion, namely, the product of the mass and the first power of the velocity. This contrariety of opinion in respect to the estimation of Moving Force, gave rise to one of the most memorable controversies in the annals of philosophy; almost all the mathematicians of Europe ultimately arranging themselves as partizans, either of the Cartesian or of the Leibnitzian doctrine. Among the adherents of Leibnitz may be mentioned John and Daniel Bernoulli, Poleni, Wolff, 'sGravesande, Camus, Muschenbroek, Papin, Hermann, Bulfinger, Kœnig, and eventually Madame du Châtelet; while in the opposite ranks may be named Maclaurin, Clarke, Stirling, Desaguliers, Catalan, Robins, Mairan, and Voltaire. The Vis Motrix, or, as Leibnitz

expressed it, the Vis Viva of a moving body was regarded as a power inherent in the body, by which it is able to encounter a certain amount of resistance before losing the whole of its velocity: the question reduced itself, therefore, to the determination of an appropriate measure of this amount of resistance, to which the Moving Force was supposed to be proportional. Leibnitz regarded the product of the mass of the body and the space through which it must move, under the action of a given retarding force, to lose the whole of its velocity, as the correct measure of the whole resistance expended in the destruction of its motion, and therefore as a proper representative of the Vis Motrix or Vis Viva of the body. Now, by the theory of uniform acceleration, $mv^2 = 2mfs$, m being the mass of the body, and s the space which it must describe, under the action of a constant retarding force f, to lose the whole of its velocity v: hence it is evident that, according to the doctrine of Leibnitz, mv^2 will represent the body's Vis Viva. On the other hand, the Cartesians estimated the whole resistance necessary for the destruction of the body's velocity by the product of the mass of the body and the whole time of the action of the given retarding force; and therefore, by the formula $mv = mft$, it would follow that mv is the proper measure of the Vis Motrix, or, in the language of Leibnitz, of the Vis Viva of the body. The memorable controversy of the Vis Viva, after raging for the space of about thirty years, was finally set to rest by the luminous observations of D'Alembert in the preface to his *Dynamique*, who declared the whole dispute to be a mere question of terms, and as having no possible connection with the fundamental principles of Mechanics. Since the publication of D'Alembert's work, the term Vis Viva has been used to signify merely the algebraical product of the mass of a moving body and the square of its velocity, while the words Moving Force have been universally employed, agreeably to the definition given by Newton in the *Principia*, in the signification of the product of the mass of a body and the accelerating force to which it is conceived to be subject, no physical theory whatever in regard to the absolute nature of force being supposed to be involved in these definitions. For additional information

respecting the controversy of the Vis Viva, the reader is referred to Montucla's *Histoire des Mathematiques*, Tom. III.; Hutton's *Mathematical Dictionary* under the word Force; and Whewell's *History of the Inductive Sciences*.

The Principle of the Conservation of Vis Viva is comprehended in the following proposition: *If a system of particles, any number of which are rigidly connected together, move from one position to another, either with or without constraint, under the action of finite accelerating forces, external or internal; the change of the vis viva of the whole system will be independent of the actions of the particles arising from their mutual connections, and will be equal to the sum of the changes which would be experienced by the vires vivæ of the particles, were each to move unconnectedly from its original to its new position through a thin smooth fixed tube, under the action of the very accelerating forces to which it is subject in the actual state of the motion.* This Principle immediately furnishes us with a first integral of the differential equations of motion, which is frequently of great use; especially if the co-ordinates of the position of the moving system involve only one independent variable, as in the problem of the Centre of Oscillation, when the Principle is sufficient for the complete determination of the motion.

The Principle employed by Huyghens[1] as the basis of his investigations on the problem of the Centre of Oscillation, constitutes under an indirect form a particular instance of the Principle of the Conservation of Vis Viva. John Bernoulli[2], however, was the first who enunciated the theory of the Conservation of Vis Viva, a name which he gave to the Principle, as a general law of nature, from which he deduced that of Huyghens as a particular case. Daniel Bernoulli[3] afterwards extended the application of the Principle to the motion of bodies subject to

[1] Si pendulum è pluribus ponderibus compositum, atque è quiete dimissum, partem quamcunque oscillationis integræ confecerit, atque inde porro intelligantur pondera ejus singula, relicto communi vinculo, celeritates acquisitas sursum convertere, ac quousque possunt ascendere; hoc facto, centrum gravitatis ex omnibus compositæ, ad eandem altitudinem reversum erit, quam ante inceptam oscillationem obtinebat. *Horolog. Oscillator*. p. 126.

[2] *Opera*, pássim.

[3] *Mémoires de l'Académie des Sciences de Berlin*, 1748.

mutual attraction, or solicited towards fixed centres by forces varying as any functions of the distances. A demonstration of the Principle in particular cases was first given by D'Alembert[1] by the aid of his general Principle of Dynamics, the same method of proof being, it was evident, of general application.

(1) A uniform rod AB (fig. 218) moves in a vertical plane, within a hemisphere; to determine its angular velocity in any of its positions, its initial position being one of instantaneous rest.

Let O be the centre of the sphere; G the middle point of AB, which will be its centre of gravity; GH a perpendicular from G upon the horizontal radius through O, which is in the plane of the rod's motion; let $OG = c$, $AG = BG = a$, $k =$ the radius of gyration about G; $OH = x$, $GH = y$, and $\theta =$ the angle of inclination of AB to the horizon at any time t. Then, by the Principle of the Conservation of Vis Viva, m being the mass of the rod,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C + 2mgy;$$

let h be the initial value of y; then, since $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, are initially zero, we have $\qquad 0 = C + 2mgh;$

hence
$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2} = 2g(y - h).$$

But from the geometry it is plain that

$$x = c\sin\theta, \qquad y = c\cos\theta,$$

whence
$$\frac{dx}{dt} = c\cos\theta\frac{d\theta}{dt}, \qquad \frac{dy}{dt} = -c\sin\theta\frac{d\theta}{dt};$$

we have, therefore,

$$(c^2 + k^2)\frac{d\theta^2}{dt^2} = 2cg(\cos\theta - \cos\alpha),$$

α being the initial value of θ; hence, putting for k^2 its value $\frac{1}{3}a^2$, we have, for the angular velocity of the rod in any of its positions,

$$\frac{d\theta^2}{dt^2} = \frac{6cg}{3c^2 + a^2}(\cos\theta - \cos\alpha).$$

[1] *Traité de Dynamique, Seconde Partie*, chap. IV. p. 252.

(2) A rod PQ (fig. 219) is kept in a vertical position by means of two small rings A and A'; its lower end P is supported on an inclined plane BC, which is at liberty to move freely on a horizontal plane; to determine the motion of the rod and the plane.

Produce QP to meet the horizontal plane in the point O; let $OP = y$, $OB = x$, at any time of the motion; $h =$ the initial value of y, $\alpha =$ the inclination of the inclined plane to the vertical, $m =$ the mass of the rod, $m' =$ the mass of the inclined plane.

Then, by the Principle of the Conservation of Vis Viva,

$$m' \frac{dx^2}{dt^2} + m \frac{dy^2}{dt^2} = C - 2gmy;$$

but, supposing the rod and the plane to be initially in a state of instantaneous rest,

$$0 = C - 2gmh;$$

hence $$m' \frac{dx^2}{dt^2} + m \frac{dy^2}{dt^2} = 2gm (h - y);$$

but, from the geometry,

$$x = y \tan \alpha, \quad \frac{dx}{dt} = \tan \alpha \frac{dy}{dt};$$

hence we have

$$(m' \tan^2 \alpha + m) \frac{dy^2}{dt^2} = 2mg (h - y),$$

$$(m' \tan^2 \alpha + m)^{\frac{1}{2}} \frac{dy}{(h - y)^{\frac{1}{2}}} = -(2mg)^{\frac{1}{2}} dt,$$

the negative sign being taken, because y decreases as t increases: therefore, by integration,

$$2 (m' \tan^2 \alpha + m)^{\frac{1}{2}} (h - y)^{\frac{1}{2}} = C + (2mg)^{\frac{1}{2}} t;$$

but $y = h$ when $t = 0$; and therefore $C = 0$; hence

$$2 (m' \tan^2 \alpha + m) (h - y) = mgt^2,$$

and therefore, for the value of y at any instant of the motion,

$$y = h - \frac{\frac{1}{2} mgt^2}{m + m' \tan^2 \alpha};$$

and therefore, for the value of x,

$$x = h \tan\alpha - \frac{\frac{1}{2} mgt^2 \tan\alpha}{m + m' \tan^2\alpha}.$$

(3) AB (fig. 220) is a uniform beam, capable of moving freely about a hinge A; the extremity B rests upon an inclined plane CE, which forms the upper surface of a body ECD; the body rests with a flat base upon a smooth horizontal plane passing through A, the vertical plane which contains AB being supposed to cut the plane surface of the body CED at right angles, and to pass through its centre of gravity: to determine the motion of the beam and the body.

Let G be the centre of gravity of AB; draw GH at right angles to the straight line ACD; let m, m', denote the masses of the beam and of the body; let $AH = x$, $GH = y$, $\angle BAC = \theta$, $\angle ECD = \alpha$, $AC = x'$, $k =$ the radius of gyration of AB about G. Then, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) + mk^2 \frac{d\theta^2}{dt^2} + m' \frac{dx'^2}{dt^2} = C - 2mgy:$$

but from the geometry we see that

$$x = a\cos\theta, \quad y = a\sin\theta, \quad x' = \frac{2a}{\sin\alpha}\sin(\alpha - \theta)\ldots\ldots(1);$$

hence we have

$$m(a^2 + k^2)\frac{d\theta^2}{dt^2} + \frac{4a^2}{\sin^2\alpha} m' \cos^2(\alpha - \theta)\frac{d\theta^2}{dt^2} = C - 2mga\sin\theta,$$

$$\{m(a^2 + k^2)\sin^2\alpha + 4m'a^2\cos^2(\alpha - \theta)\}\frac{d\theta^2}{dt^2} = \sin^2\alpha\,(C - 2mga\sin\theta);$$

let β be the value of θ when $\frac{d\theta}{dt} = 0$; then

$$0 = \sin^2\alpha\,(C - 2mga\sin\beta),$$

and therefore we get

$$\{m(a^2 + k^2)\sin^2\alpha + 4m'a^2\cos^2(\alpha - \theta)\}\frac{d\theta^2}{dt^2} = 2mag\sin^2\alpha\,(\sin\beta - \sin\theta),$$

which gives the value of $\frac{d\theta}{dt}$ for any assigned value of θ; whence,

by the aid of the equations (1), we may obtain the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dx'}{dt}$, for any position of the beam.

(4) A uniform lever ACB, (fig. 221), of which the arms AC and BC are at right angles to each other, rests in equilibrium when AC is inclined at a given angle to the horizon; if AC be raised to a horizontal position, C being fixed, to find the angle through which it will fall.

Let $CA = 2a$, $CB = 2a'$, $m =$ the mass of AC, $m' =$ the mass of BC; let θ, θ', be the inclinations of CA, CB, to the horizon, at any time of the motion.

Then the vis viva of the lever will be equal to

$$2mag \sin\theta + 2m'a'g \sin\theta' + C;$$

but, when $\theta = 0$ and therefore $\theta' = \frac{1}{2}\pi$, the vis viva is equal to zero; hence

$$0 = 2m'a'g + C;$$

hence the vis viva for any position of the lever is equal to

$$2mag \sin\theta + 2m'a'g \sin\theta' - 2m'a'g.$$

Now, when the value of θ is a maximum, the vis viva will again become zero; hence, for the required value of θ,

$$ma \sin\theta + m'a' \sin\theta' = m'a' \quad\quad (1).$$

Let β, β', be the values of θ, θ', for the equilibrium of the lever; then

$$ma \cos\beta = m'a' \cos\beta';$$

hence from (1) there is

$$\cos\beta' \sin\theta + \cos\beta \sin\theta' = \cos\beta,$$

or, since $\beta' = \frac{1}{2}\pi - \beta$, $\quad \theta' = \frac{1}{2}\pi - \theta$,

$$\sin\beta \sin\theta + \cos\beta \cos\theta = \cos\beta, \quad \cos(\theta - \beta) = \cos\beta;$$

and therefore $\theta = 2\beta$, the angle through which CA falls.

(5) To determine the motion of a pendulum, the axis of which is a cylinder resting upon two perfectly rough planes which coincide with the same horizontal plane, the cylindrical axis being thus capable of rolling along the planes.

Let C (fig. 222) be the centre of a vertical section of the cylindrical axis made by a plane containing the centre of gravity of the pendulum; C may be regarded as the centre of gravity of the axis. Let G be the centre of gravity of the pendulum and cylinder together, and mk^2 their moment of inertia about a horizontal line through G parallel to the axis, m denoting the sum of their masses. Let GH be drawn at right angles to the horizontal plane along which the axis rolls; let O be the point of contact of the section C of the axis with this plane at any time of the motion, A being the position of O corresponding to the equilibrium of the system. Let $CO = c$, $CG = a$, $\angle\, GCK = \phi$, CK being a vertical line, $AH = x$, $GH = y$.

Then the vis viva of the system at the time t due to the motion of G will be $m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)$, and the vis viva due to rotation about G will be $mk^2 \frac{d\phi^2}{dt^2}$; hence the whole vis viva of the system will be equal to

$$m\left(k^2 \frac{d\phi^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right):$$

also the sum of the products of the mass of each molecule of the system into the vertical space through which it has descended, will be equal to my together with some constant quantity depending upon the initial circumstances of the system. Hence, by the Principle of Conservation of Vis Viva,

$$m\left(k^2 \frac{d\phi^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) = C + 2gmy;$$

but from the geometry it is evident that

$$x = a \sin\phi - c\phi, \quad y = a \cos\phi - c,$$

and therefore

$$\frac{dx}{dt} = (a \cos\phi - c)\frac{d\phi}{dt}, \quad \frac{dy}{dt} = -a \sin\phi \frac{d\phi}{dt};$$

hence we have

$$(a^2 + c^2 + k^2 - 2ac \cos\phi)\frac{d\phi^2}{dt^2} = C' + 2g(a \cos\phi - c);$$

let α be the maximum value of ϕ, then

$$0 = C' + 2g\,(a \cos \alpha - c),$$

and therefore

$$(a^2 + c^2 + k^2 - 2ac \cos \phi)\,\frac{d\phi^2}{dt^2} = 2ga\,(\cos \phi - \cos \alpha),$$

which gives the angular velocity of the pendulum for every position which it assumes during its motion.

For the period of a semi-oscillation we have

$$T = \frac{1}{(2ag)^{\frac{1}{2}}} \int_0^{\alpha} \frac{(a^2 + c^2 + k^2 - 2ac \cos \phi)^{\frac{1}{2}}}{(\cos \phi - \cos \alpha)^{\frac{1}{2}}}\, d\phi \ldots\ldots\ldots\ldots (1).$$

This integration cannot be effected except, which we will suppose to be the case, the amplitude of the pendulum's oscillation is very small.

Assume then $\sin \dfrac{\phi}{2} = s, \quad \sin \dfrac{\alpha}{2} = b;$

whence $\cos \phi = 1 - 2 \sin^2 \dfrac{\phi}{2} = 1 - 2s^2,$

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2} = 1 - 2b^2,$$

and $d\phi = \dfrac{4sds}{\sin \phi} = \dfrac{4sds}{(1 - \cos^2 \phi)^{\frac{1}{2}}} = \dfrac{2ds}{(1 - s^2)^{\frac{1}{2}}}.$

Hence, from (1),

$$T = \frac{1}{(2ag)^{\frac{1}{2}}} \int_0^{b} \frac{(a^2 + c^2 + k^2 - 2ac + 4ac\, s^2)^{\frac{1}{2}}}{(2b^2 - 2s^2)^{\frac{1}{2}}} \cdot \frac{2ds}{(1 - s^2)^{\frac{1}{2}}},$$

and therefore, putting $(a - c)^2 + k^2 = h^2$,

$$T = \frac{1}{(ag)^{\frac{1}{2}}} \int_0^{b} \frac{(h^2 + 4ac\, s^2)^{\frac{1}{2}}\, ds}{(1 - s^2)^{\frac{1}{2}} (b^2 - s^2)^{\frac{1}{2}}};$$

but, s being a small quantity, we have, neglecting small quantities of orders higher than the second,

$$\frac{1}{(1 - s^2)^{\frac{1}{2}}} = (1 - s^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2} s^2,$$

$$(h^2 + 4ac\, s^2)^{\frac{1}{2}} = h\left(1 + \frac{2ac}{h^2} s^2\right),$$

and therefore

$$\frac{(k^2 + 4ac\,s^2)^{\frac{1}{2}}}{(1-s^2)^{\frac{1}{2}}} = k\left(1 + \frac{4ac + k^2}{2k^2}s^2\right) \quad \text{nearly:}$$

hence

$$T = \frac{k}{(ag)^{\frac{1}{2}}}\int_0^b ds\left\{\frac{1}{(b^2-s^2)^{\frac{1}{2}}} + \frac{4ac+k^2}{2k^2}\,\frac{s^2}{(b^2-s^2)^{\frac{1}{2}}}\right\}$$

$$= \frac{k}{(ag)^{\frac{1}{2}}}\left\{\frac{\pi}{2} + \frac{4ac+k^2}{2k^2}\,\frac{\pi b^2}{4}\right\}$$

$$= \frac{\pi k}{2\,(ag)^{\frac{1}{2}}} + \frac{\pi b^2\,(4ac+k^2)}{8k\,(ag)^{\frac{1}{2}}};$$

and therefore the period of a complete oscillation is equal to

$$\frac{\pi k}{(ag)^{\frac{1}{2}}} + \frac{\pi b^2\,(4ac+k^2)}{4k\,(ag)^{\frac{1}{2}}}.$$

Euler has discussed this problem, starting with the general equations of motion, and investigated the pressure on the plane at any time, as well as the horizontal action of the plane upon the cylinder which shall be sufficient to prevent sliding.

Euler; *Nova Acta Acad. Petrop.* 1788; p. 145.

(6) Two equal particles are attached to the extremities A, A', (fig. 223), of a straight lever ACA' having equal arms without weight, and are each attracted to a centre of force in O which varies inversely as the cube of the distance; CO is vertical and equal to CA or CA'; supposing the lever to be placed originally in a given position, to find the time of its becoming vertical.

Let $CO = CA = a$; $OA = r$ and $OA' = r'$ at any time of the motion; $\angle ACO = \theta$; $m =$ the mass of each of the particles, $\mu =$ the attraction of the force in O upon a unit of matter collected in a point at a unit of distance; $\alpha =$ the initial value of θ.

Then the vis viva of the two particles together will be, at any time t, $2ma^2\frac{d\theta^2}{dt^2}$; hence

$$2ma^2\frac{d\theta^2}{dt^2} = \int\left(-\frac{\mu m}{r^3}\,dr - \frac{\mu m}{r'^3}\,dr'\right) + C,$$

and therefore

$$2a^2\frac{d\theta^2}{dt^2}=\frac{\mu}{r^2}+\frac{\mu}{r'^2}+C'.$$

Now from the geometry we have

$$r^2=2a^2(1-\cos\theta),\qquad r'^2=2a^2(1+\cos\theta);$$

hence
$$2a^2\frac{d\theta^2}{dt^2}=\frac{\mu}{2a^2}\left(\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}\right)+C'$$
$$=\frac{\mu}{a^2}\,\frac{1}{\sin^2\theta}+C';$$

but when $\theta=\alpha$, $\frac{d\theta}{dt}=0$; hence

$$0=\frac{\mu}{a^2}\,\frac{1}{\sin^2\alpha}+C',$$

and therefore

$$2a^2\frac{d\theta^2}{dt^2}=\frac{\mu}{a^2\sin^2\alpha}\,\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\theta};$$

hence, $\frac{d\theta}{dt}$ being negative because θ decreases with the increase of t,

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\frac{\sin\theta\,d\theta}{(\sin^2\alpha-\sin^2\theta)^{\frac{1}{2}}}=-\frac{dt}{\sin\alpha},$$

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\frac{d\cos\theta}{(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}}=\frac{dt}{\sin\alpha};$$

integrating we get

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\{\cos\theta+(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}\}=\frac{t}{\sin\alpha}+C;$$

but $\theta=\alpha$ when $t=0$; hence $C=\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\cos\alpha$; and therefore

$$\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\log\frac{\cos\theta+(\cos^2\theta-\cos^2\alpha)^{\frac{1}{2}}}{\cos\alpha}=\frac{t}{\sin\alpha}.$$

When ACA' becomes vertical, $\theta=0$, and we have for the required value of t,

$$t=\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\sin\alpha\log\frac{1+\sin\alpha}{\cos\alpha}=\left(\frac{2}{\mu}\right)^{\frac{1}{2}}a^2\sin\alpha\log\tan\frac{\pi+2\alpha}{4}.$$

(7) BFG (fig. 224) is a heavy body of any form, of which C is the centre of gravity; an inextensible string attached to a fixed point E is wound about the circumference of a circle ALH, having C for its centre, and representing an axis; EA is vertical; to determine the velocity of C when the body has descended from rest through a given altitude, under the action of gravity, by the uncoiling of the string.

Let a be the radius of the axis, k the radius of gyration of the body about C; v the velocity acquired by C, after descending through a space x. Then

$$v^2 = \frac{2ga^2x}{a^2+k^2}.$$

This problem is one of the 'Theoremata Selecta,' given by John Bernoulli, '*pro conservatione virium vivarum demonstranda et experimentis confirmanda.*'

Comment. Acad. Petrop. 1727, p. 200. *Opera*, Tom. III. p. 127.

(8) A particle A (fig. 225) descends down the curve CKA, drawing a particle B up the curve CLB by means of a string passing over the point C; to determine the velocities of the particles after moving from rest through any corresponding spaces.

Let m, m', be the masses of A, B, respectively; v, v', their velocities after moving through vertical spaces equal to y, y'; then, ds, ds', denoting elements of the two curves,

$$v^2 = 2g(my - m'y')\frac{ds^2}{mds^2 + m'ds'^2}, \quad v'^2 = 2g(my - m'y')\frac{ds'^2}{mds^2 + m'ds'^2}.$$

John Bernoulli; *Act. Erudit. Lips.* 1735. Mai. p. 210; *Opera*, Tom. III. p. 257. Hermann; *Mémoires de St. Pétersbourg*, Tom. II. D'Alembert; *Traité de Dynamique*, p. 123; Seconde Edition.

(9) A uniform straight plank rests with its middle point upon a rough horizontal cylinder, their directions being perpendicular

to each other: supposing the plank to be slightly displaced, so as to remain always in contact with the cylinder without sliding, to determine the period of one oscillation.

If $2a$ = the length of the plank, and r = the radius of the circle, the time of an oscillation is equal to

$$\frac{\pi a}{(3gr)^{\frac{1}{2}}}.$$

(10) Two equal weights P, P, are tied to the ends of a fine string which passes over two pulleys without mass in a horizontal line: supposing a weight W, less than $2P$, to be fixed to the middle point of the horizontal portion of the string, to determine how far it will descend.

If a = the distance between the two pulleys, W will fall through a space equal to

$$\frac{2PWa}{4P^2 - W^2}.$$

(11) A solid cylinder is freely moveable about its axis, which is fixed horizontally, and weights W, W', are hung at the ends of a string wound round it: after W' has descended from rest for t seconds, it is suddenly cut off, and the system comes to rest in t seconds more: to find the weight of the cylinder.

The weight of the cylinder is equal to

$$\frac{4W^2}{W' - 2W}.$$

(12) A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed: a uniform rod, such as just to fit the bore of the tube, is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube: supposing the velocity of projection to be known, to find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

If v be the velocity of the rod's projection, m the mass of the rod, m' that of the tube, $2a$, $2a'$, their respective lengths, and ω the required angular velocity; then

$$\omega^2 = \frac{3mv^2}{ma^2 + m'a'^2}.$$

SECT. 2. *Vis Viva and the Conservation of the Motion of the Centre of Gravity.*

The Principle of the Conservation of the Motion of the Centre of Gravity, under its most general form, asserts that, *the motion of the centre of gravity of a free system of bodies disposed relatively to each other in any conceivable manner, is always the same as if the bodies were all united in the centre of gravity, and at the same time each of them were animated by the same accelerating forces as in their actual state.* The discovery of the Principle is due to Newton[1], by whom it received a demonstration in the particular case where the system is subject to no external force, when the centre of gravity will either remain at rest or move in a straight line with a uniform velocity. D'Alembert[2] afterwards extended the Principle to the case where each body is supposed to be solicited by a constant accelerating force acting in parallel lines, or directed towards a fixed point and varying as the distance. Finally, Lagrange[3] expressed the Principle under its most general form for every law of force to which the bodies can be subject.

(1) A smooth groove KAL (fig. 226) is carved in a vertical plane in the body $KBCL$, which is placed upon a smooth horizontal plane, along which it is able to slide freely; to find the form of the groove that a heavy particle, placed within it, may oscillate in it tautochronously, the time of an oscillation being given.

Let P be the place of the particle in the groove at any time; draw PN vertically to meet the horizontal plane in N, which

[1] *Principia; Axiomata sive Leges Motus*, Cor. 4.
[2] *Traité de Dynamique, Seconde Partie*, Chap. II.
[3] *Mécanique Analytique*, Tom. I. p. 257, &c.

will lie in the line OE formed by the intersection of a vertical plane through the groove with the horizontal plane. Let A be the lowest point of the groove, draw AM horizontally, AA' vertically. Let O be a fixed point in OE; let $OA' = x'$, $ON = x_1$, $PN = y_1$, $AM = x$, $PM = y$; let k_1, k, be the initial values of y_1, y; let m = the mass of the particle, m' = the mass of the body.

Then, by the Principle of the Conservation of the Motion of the Centre of Gravity, since no forces act upon the particle and body parallel to OE,

$$m'\frac{dx'}{dt} + m\frac{dx_1}{dt} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, by the Principle of the Conservation of Vis Viva,

$$m'\frac{dx'^2}{dt^2} + m\left(\frac{dx_1^2}{dt^2} + \frac{dy_1^2}{dt^2}\right) = 2mg\,(k_1 - y_1) \quad\ldots\ldots\ldots(2).$$

But, from the geometry, it is evident that

$$\frac{dx_1}{dt} = \frac{dx'}{dt} + \frac{dx}{dt} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and
$$k_1 - y_1 = k - y, \qquad \frac{dy_1}{dt} = \frac{dy}{dt} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (1) and (3) we have

$$\frac{dx_1}{dt} = \frac{m'}{m+m'}\frac{dx}{dt}, \qquad \frac{dx'}{dt} = -\frac{m}{m+m'}\frac{dx}{dt} \quad\ldots\ldots\ldots(5).$$

Hence, from (2), (4), (5), we see that

$$\frac{m'}{m+m'}\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = 2g\,(k-y);$$

and therefore, if τ denote the time of a semi-oscillation,

$$\tau = -\frac{1}{(2g)^{\frac{1}{2}}}\int_k^0 \frac{\left(\dfrac{m'}{m+m'}\dfrac{dx^2}{dy^2} + 1\right)^{\frac{1}{2}}}{(k-y)^{\frac{1}{2}}}\,dy \quad\ldots\ldots\ldots\ldots(6).$$

This value of τ must be independent of k in order that the particle may oscillate tautochronously, and therefore we must

have, it being necessary that the coefficient of dy be of -1 dimensions in y and k,

$$\left(\frac{m'}{m+m'}\frac{dx^2}{dy^2}+1\right)^{\frac{1}{2}}=\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}}\text{.......................(7)},$$

where a is a constant quantity; hence

$$\frac{dx}{dy}=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left(\frac{a-y}{y}\right)^{\frac{1}{2}},$$

and therefore, by integration,

$$x=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left\{(ay-y^2)^{\frac{1}{2}}+\tfrac{1}{2}a\,\text{vers}^{-1}\frac{2y}{a}\right\}\text{.........(8)}.$$

But, from (6) and (7),

$$\tau=-\frac{1}{(2g)^{\frac{1}{2}}}\int_k^0\frac{a^{\frac{1}{2}}\,dy}{(ky-y^2)^{\frac{1}{2}}}=\frac{\pi a^{\frac{1}{2}}}{(2g)^{\frac{1}{2}}},\quad a=\frac{2g\tau^2}{\pi^2},$$

and therefore from (8) we get, for the equation to the groove,

$$x=\left(\frac{m+m'}{m'}\right)^{\frac{1}{2}}\left\{\left(\frac{2g\tau^2}{\pi^2}y-y^2\right)^{\frac{1}{2}}+\frac{g\tau^2}{\pi^2}\,\text{vers}^{-1}\frac{\pi^2 y}{g\tau^2}\right\}.$$

Clairaut; *Mémoires de l'Académie des Sciences de Paris*, 1742, p. 41. Euler; *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48.

(2) A thin spherical shell, the radius of which is a, rests upon a smooth horizontal plane; a particle of the same mass as the shell, is placed at the lowest point of its internal surface, which is smooth: to determine to what height the particle will ascend, supposing the shell to be projected with a horizontal velocity $2(ga)^{\frac{1}{2}}$.

The particle will ascend just as high as the centre of the shell, and then descend.

SECT. 3. *Vis Viva and the Conservation of Areas.*

The Principle of the Conservation of Areas asserts, that *if a system of particles be subject only to mutual actions, the sum of the products of the mass of each particle into the projection* (on

any proposed plane) of the area described by its radius vector round any assigned point, is proportional to the time. The same principle holds good also if the system be subject to external forces, provided that they be such that the algebraical sum of their moments about a line through the assigned point at right angles to the proposed plane be zero. This principle, which is in fact a generalization of Newton's theorem respecting the areas described by a single body about a centre of force, was discovered, about the same time, by Euler[1], Daniel Bernoulli[2], and D'Arcy[3]; the enunciation of the Principle given by Euler and Bernoulli being expressed under a form somewhat different from that given by D'Arcy, under which it is now generally expressed. The discovery of the Principle was suggested to these three mathematicians by the consideration of the problem of the motion of several bodies within a tube of given form, moving about a fixed point.

(1) P, Π, (fig. 227), are two material particles attached to an inflexible straight line $PO\Pi$, moveable in a horizontal plane about a fixed point O; the particle Π is fixed to the inflexible line, while the particle P is capable of sliding along it; to determine the path described by P, corresponding to any initial velocities of the particles.

Let OE be an immoveable straight line passing through O; let $PO = r$, $\Pi O = a$, $m =$ the mass of P, $\mu =$ the mass of Π, $\angle POE = \theta$. Then, by the Principle of the Conservation of Areas, since the only force to which the moving system is subject is the reaction of the fixed point O, we have

$$(mr^2 + \mu a^2)\frac{d\theta}{dt} = C \text{.......................} (1),$$

where C is some constant quantity.

Again, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2}\right) + \mu a^2\frac{d\theta^2}{dt^2} = C',$$

[1] *Opuscula, de motu corporum tubis mobilibus inclusorum*, p. 48, 1746.

[2] *Mémoires de l'Académie des Sciences de Berlin*, 1745, p. 54.

[3] *Mémoires de l'Académie des Sciences de Paris*, 1747, p. 348.

$$m\frac{dr^2}{dt^2} + (mr^2 + \mu a^2)\frac{d\theta^2}{dt^2} = C' \text{.............} (2),$$

C' being a constant quantity.

Eliminating dt between (1) and (2), we obtain

$$m\frac{dr^2}{d\theta^2} + mr^2 + \mu a^2 = \frac{C'}{C^2}(mr^2 + \mu a^2)^2,$$

$$m\frac{dr^2}{d\theta^2} = \left\{\frac{C'}{C^2}(mr^2 + \mu a^2) - 1\right\}(mr^2 + \mu a^2),$$

which is the differential equation to P's path.

In order to determine C and C', suppose that b, ω, u, are the initial values of r, $\frac{d\theta}{dt}$, $\frac{dr}{dt}$, respectively. Then, from (1),

$$(mb^2 + \mu a^2)\,\omega = C,$$

which determines C; and, from (2),

$$mu^2 + (mb^2 + \mu a^2)\,\omega^2 = C',$$

which determines C'.

Clairaut; *Mém. de l'Acad. des Sciences de Paris*, 1742, p. 22. D'Arcy: *Mém. de l'Acad. des Sciences de Paris*, 1747, p. 351. D'Alembert; *Traité de Dynamique*, p. 104, seconde edit.

(2) A straight rod PQ, (fig. 228), subject to the condition of always passing through a small fixed ring at O, is in motion on a horizontal plane; to determine the path of its centre of gravity G.

At any time t of the motion, let $OG = r$, $\angle GOE = \theta$, OE being a fixed line in the plane. Let m be the mass of an element of the rod at any distance ρ from O, and let μ be the mass of the whole rod.

Then, by the Principle of the Conservation of Areas, the only force which acts on the rod being the reaction of the ring,

$$C = \Sigma\left(m\rho^2\frac{d\theta}{dt}\right) = \Sigma\,(m\rho^2)\frac{d\theta}{dt} = \mu\,(r^2 + k^2)\frac{d\theta}{dt} \text{...........} (1),$$

k being the radius of gyration of the rod about its centre of gravity, and C a constant quantity.

Again, by the Principle of the Conservation of Vis Viva, the ring being considered perfectly smooth,

$$C' = \mu\left(\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2}\right) + \mu k^2\frac{d\theta^2}{dt^2}$$

$$= \mu\frac{dr^2}{dt^2} + \mu(r^2+k^2)\frac{d\theta^2}{dt^2} \quad\text{..................}(2),$$

C' being a constant quantity.

Eliminating dt between (1) and (2), we have

$$\frac{dr^2}{d\theta^2} + r^2 + k^2 = \frac{\mu C'}{C^2}(r^2+k^2)^2,$$

$$\frac{dr^2}{d\theta^2} = \left\{\frac{\mu C'}{C^2}(r^2+k^2) - 1\right\}(r^2+k^2) \quad\text{.............}(3).$$

Suppose that a, u, ω, are the initial values of r, $\frac{dr}{dt}$, $\frac{d\theta}{dt}$, respectively; then, by (1) and (2),

$$C = \mu(a^2+k^2)\,\omega, \qquad C' = \mu u^2 + \mu(a^2+k^2)\,\omega^2:$$

hence the equation (3) becomes

$$\frac{dr^2}{d\theta^2} = \left\{\frac{u^2+(a^2+k^2)\,\omega^2}{(a^2+k^2)^2\,\omega^2}(r^2+k^2) - 1\right\}(r^2+k^2)$$

$$= \frac{r^2+k^2}{\omega^2(a^2+k^2)^2}\left\{(r^2+k^2)\,u^2 + \omega^2(a^2+k^2)(r^2-a^2)\right\},$$

which is the differential equation to the path of G.

Clairaut; *Mémoires de l'Acad. des Sciences de Paris*, 1742, p. 38–41.

(3) Two equal particles P, P, (fig. 229), are attached to the extremities of a rod PP; the middle point O of the rod is fixed; the rod is able to move in every direction about O; to determine the motion of the particles corresponding to any initial circumstances, the weight of the rod being neglected.

Through the point O draw a straight line AOB; with O as a centre and radius equal to OP, describe the two indefinite

circular arcs APk, Al, the latter of which is supposed to lie within an assigned plane. Let $OP=a$, $\angle AOP=\phi$, $\angle kAl=\theta$; $m=$ the mass of each of the particles. Then, t denoting the corresponding time, we shall have, by the Principle of the Conservation of Vis Viva, whether the particles be subject to the action of gravity or not,

$$2ma^2\left(\frac{d\phi^2}{dt^2}+\sin^2\phi\,\frac{d\theta^2}{dt^2}\right)=C,$$

or

$$\frac{d\phi^2}{dt^2}+\sin^2\phi\,\frac{d\theta^2}{dt^2}=c\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

C, c, being constant quantities.

Again, by the Principle of the Conservation of Areas, we have

$$2ma^2\sin^2\phi\,\frac{d\theta}{dt}=C_1,$$

or

$$\sin^2\phi\,\frac{d\theta}{dt}=c_1\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

C_1, c_1, being constants.

Eliminating $\frac{d\theta}{dt}$ between the equations (1) and (2) we get

$$\frac{d\phi^2}{dt^2}+\frac{c_1^2}{\sin^2\phi}=c,$$

and therefore $\quad \sin\phi\,d\phi=(c\sin^2\phi-c_1^2)^{\frac{1}{2}}\,dt,$

$$(c-c_1^2-c\cos^2\phi)^{\frac{1}{2}}\,dt=-d\cos\phi;$$

integrating, and adding an arbitrary constant c_2,

$$t+c_2=\frac{1}{c^{\frac{1}{2}}}\cos^{-1}\frac{c^{\frac{1}{2}}\cos\phi}{(c-c_1^2)^{\frac{1}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\cos\phi=\frac{(c-c_1^2)^{\frac{1}{2}}}{c^{\frac{1}{2}}}\cos\{c^{\frac{1}{2}}(t+c_2)\}\,\ldots\ldots\ldots(4)$$

Suppose that when $t=0$, $\phi=\beta$, $\frac{d\theta}{dt}=\omega$, $\frac{d\phi}{dt}=\omega'$; then

$$c=\omega'^2+\omega^2\sin^2\beta,\quad\text{from (1)};$$

$$c_1=\omega\sin^2\beta,\quad\text{from (2)};$$

and therefore, from (3),

$$c_2 = \frac{1}{(\omega'^2 + \omega^2 \sin^2\beta)^{\frac{1}{2}}} \cos^{-1} \frac{(\omega'^2 + \omega^2 \sin^2\beta)^{\frac{1}{2}} \cos\beta}{(\omega'^2 + \omega^2 \sin^2\beta \cos^2\beta)^{\frac{1}{2}}}.$$

The value of $\cos\phi$ being given by (4), we may then, by the aid of (2), get an expression for θ in terms of t.

(4) A spherical shell, the interior radius of which is the n^{th} of the exterior, is filled with fluid of the same density with itself; to compare the space through which it would roll from rest in a given time down a perfectly rough inclined plane, with that which would be described by a solid sphere of the same size and weight rolling down the same plane.

Let m, m', denote the masses of the shell and fluid respectively; a, a', the exterior and interior radii; k, k', the radii of gyration of the shell and fluid about a diameter of the sphere; α the inclination of the plane to the horizon; θ, θ', the angles through which the shell and fluid have revolved about their common centre of gravity from the beginning of the motion; x the space described by the centre of the sphere.

Then, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + k^2 \frac{d\theta^2}{dt^2}\right) + m'\left(\frac{dx^2}{dt^2} + k'^2 \frac{d\theta'^2}{dt^2}\right) = C + 2(m + m')\, gx \sin\alpha;$$

but, since the resultant of all the forces which act on the fluid-sphere passes through its centre of gravity, we have, by the Principle of the Conservation of Areas,

$$m'k'^2 \frac{d\theta'}{dt} = C';$$

hence, from these two equations,

$$m\left(\frac{dx^2}{dt^2} + k^2 \frac{d\theta^2}{dt^2}\right) + m' \frac{dx^2}{dt^2} = C'' + 2(m + m')\, gx \sin\alpha;$$

but, since the sphere rolls, we have $x = a\theta$; hence, putting $m + m' = \mu$,

$$(\mu a^2 + mk^2) \frac{dx^2}{dt^2} = C''a^2 + 2\mu a^2 gx \sin\alpha;$$

differentiating with respect to t, and dividing by $2\dfrac{dx}{dt}$,

$$(\mu a^2 + mk^2)\frac{d^2x}{dt^2} = \mu a^2 g \sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, $\quad mk^2 = \frac{2}{5}\mu a^2 - \frac{2}{5}m'a'^2 = \frac{2}{5}(\mu a^2 - m'a'^2)$;

but $m' = \dfrac{\mu}{n^3}$ and $a' = \dfrac{a}{n}$; hence

$$mk^2 = \tfrac{2}{5}\mu a^2\frac{n^5-1}{n^5}.$$

Substituting this value of mk^2 in (1), we obtain

$$\left(1 + \tfrac{2}{5}\frac{n^5-1}{n^5}\right)\frac{d^2x}{dt^2} = g\sin\alpha,$$

$$(7n^5 - 2)\frac{d^2x}{dt^2} = 5n^5 g\sin\alpha;$$

integrating, and bearing in mind that $\dfrac{dx}{dt} = 0$, $x = 0$, when $t = 0$, we get

$$(7n^5 - 2)\,x = \tfrac{5}{2}n^5 g t^2 \sin\alpha.$$

In the case of the solid sphere, we shall obtain in a similar manner,

$$7x' = \tfrac{5}{2}gt^2\sin\alpha,$$

x' denoting the space through which it has rolled along the plane at the end of the time t.

Hence finally we get $\quad \dfrac{x}{x'} = \dfrac{7n^5}{7n^5 - 2}$.

If $n = 2$, we have $\quad \dfrac{x}{x'} = \dfrac{112}{111}$.

Lady's and Gentleman's Diary, 1842, p. 51.

(5) The particle m (fig. 230) is connected with the particles m' and m'' by means of two fine inextensible strings mOm', mOm'', passing through a small smooth ring at O; m, m', m'', all lie on a smooth horizontal plane passing through O; to determine the tensions of the two strings and the motions of the particles, supposing the particles to have received any initial impulses such

as, at least at the commencement of the motion, to keep the strings at full stretch.

Draw through O a straight line OA in the plane of the motions; let $Om=r$, $Om'=r'$, $Om''=r''$, $\angle mOA=\theta$, $\angle m'OA=\theta'$, $\angle m''OA=\theta''$, at any time t; $T'=$ the tension of the string mOm', and $T''=$ that of the string mOm''.

Then, since the only forces which act upon the particles pass through O, we have, by a formula in the theory of Central Forces,

$$\left.\begin{aligned} \frac{d^2r'}{dt^2} &= r'\frac{d\theta'^2}{dt^2} - \frac{T'}{m'}, \\ \frac{d^2r''}{dt^2} &= r''\frac{d\theta''^2}{dt^2} - \frac{T''}{m''}, \\ \frac{d^2r}{dt^2} &= r\frac{d\theta^2}{dt^2} - \frac{T'+T''}{m}. \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

But, the strings being supposed to be kept at full stretch, we have

$$r+r'=c', \quad r+r''=c'', \dots\dots\dots\dots\dots\dots\dots (2);$$

where c', c'', denote the lengths of the strings mOm', mOm''; and therefore

$$\frac{d^2r}{dt^2}+\frac{d^2r'}{dt^2}=0, \qquad \frac{d^2r}{dt^2}+\frac{d^2r''}{dt^2}=0;$$

hence, by the first and third of the formulæ (1),

$$\frac{T'}{m'}+\frac{T'+T''}{m}=r\frac{d\theta^2}{dt^2}+r'\frac{d\theta'^2}{dt^2} \dots\dots\dots\dots\dots (3);$$

and, by the second and third,

$$\frac{T''}{m''}+\frac{T'+T''}{m}=r\frac{d\theta^2}{dt^2}+r''\frac{d\theta''^2}{dt^2} \dots\dots\dots\dots (4).$$

Again, since the only forces which act upon the three particles pass through the point O, we have, by the Principle of the Conservation of Areas,

$$r^2\frac{d\theta}{dt}=e, \qquad r'^2\frac{d\theta'}{dt}=e', \qquad r''^2\frac{d\theta''}{dt}=e'', \dots\dots\dots\dots (5);$$

where e, e', e'', are invariable quantities: hence, from (3) and (4), we have

$$\frac{T'}{m'}+\frac{T'+T''}{m}=\frac{e^2}{r^3}+\frac{e'^2}{r'^3},$$

$$\frac{T''}{m''}+\frac{T'+T''}{m}=\frac{e^2}{r^3}+\frac{e''^2}{r''^3};$$

from these two equations we may readily ascertain that

$$\left.\begin{aligned}(m+m'+m'')\frac{T'}{m'}&=\frac{me^2}{r^3}+\frac{(m+m'')e'^2}{r'^3}-\frac{m''e''^2}{r''^3},\\(m+m'+m'')\frac{T''}{m''}&=\frac{me^2}{r^3}+\frac{(m+m')e''^2}{r''^3}-\frac{m'e'^2}{r'^3},\\(m+m'+m'')\frac{T'+T''}{m}&=\frac{(m'+m'')e^2}{r^3}+\frac{m'e'^2}{r'^3}+\frac{m''e''^2}{r''^3},\end{aligned}\right\}\ldots\ldots(6),$$

which give the values of the tensions of the two strings mOm', mOm'', and of the double string Om. It is important to observe that these values for the tensions hold good only so long as both the strings are at full stretch; if either of the strings become slack at any epoch of the motion, these formulæ will no longer apply; this will be evident when it is considered that in obtaining them we made use of the equations (2) which are grounded on the supposition that the strings are at full stretch. The formulæ themselves will indicate the epoch at which their inapplicability may commence by giving a zero value for either T' or T''.

Again, by the Principle of Vis Viva,

$$m\left(r^2\frac{d\theta^2}{dt^2}+\frac{dr^2}{dt^2}\right)+m'\left(r'^2\frac{d\theta'^2}{dt^2}+\frac{dr'^2}{dt^2}\right)+m''\left(r''^2\frac{d\theta''^2}{dt^2}+\frac{dr''^2}{dt^2}\right)=C,$$

where C is some constant quantity: hence, observing that, by the equations (2), $\frac{dr'}{dt}=-\frac{dr}{dt}=\frac{dr''}{dt}$, we get

$$mr^2\frac{d\theta^2}{dt^2}+m'r'^2\frac{d\theta'^2}{dt^2}+m''r''^2\frac{d\theta''^2}{dt^2}+(m+m'+m'')\frac{dr^2}{dt^2}=C,$$

and therefore, by the equations (5),

$$\frac{me^2}{r^2}+\frac{m'e'^2}{r'^2}+\frac{m''e''^2}{r''^2}+(m+m'+m'')\frac{dr^2}{d\theta^2}\frac{e^2}{r^4}=C\ldots\ldots\ldots(7),$$

and thence, by the equations (2), putting $m+m'+m''=\mu$,

$$\frac{me^2}{r^2}+\frac{m'e'^2}{(c'-r)^2}+\frac{m''e''^2}{(c''-r)^2}+\frac{\mu e^2}{r^4}\frac{dr^2}{d\theta^2}=C\ldots\ldots\ldots\ldots\ldots(8),$$

which is the differential equation to the path of m. Similarly may be obtained the differential equations to the paths of m' and m''. These equations will evidently cease to define the paths of the particles if at any time either of the strings become slack, or either T' or T'' become zero. If either of the strings become slack at any time, then we shall have to investigate the motions of the two particles whose connecting string is not slack, the particle which belongs to the loose string moving along for a time without constraint. From the equation (7) it is evident that none of the quantities r, r', r'', can ever become zero; or that the particles, so long as the strings are tight, will none of them arrive at the point O.

Suppose that ω, ω', ω'', a, β, are the initial values of $\frac{d\theta}{dt}$, $\frac{d\theta'}{dt}$, $\frac{d\theta''}{dt}$, r, $\frac{dr}{dt}$; then, from the equations (5),

$$e = a^2\omega, \qquad e' = (c' - a)^2\omega', \qquad e'' = (c'' - a)^2\omega'';$$

which give the values of e, e', e'': and then, from (8),

$$C = ma^2\omega^2 + m'(c' - a)^2\omega'^2 + m''(c'' - a)^2\omega''^2 + \mu\beta^2.$$

If instead of attaching two particles m', m'', to m, we had attached any number of them, the problem would have been essentially of no greater difficulty.

Riccati; *Comment. Bonon.* Tom. v. P. I. p. 150; anno 1767.

(6) The bob of a pendulum is a hollow sphere, smooth internally, which can be filled with a fluid or with a solid sphere, fixed to the bob, of the same density as the fluid: to find the length of the equivalent simple pendulum, (1) when the cavity is filled with the solid, (2) when it is filled with the fluid, the rod and cavity being supposed to be rigid and without weight.

Let mk^2 = the moment of inertia of the solid or fluid sphere about a diameter, a = the distance of the centre of the sphere from the point of suspension, r = the radius of the sphere, θ = the inclination of the rod to the vertical at any time t, and ω = the angular velocity of the sphere about a diameter parallel to the axis of suspension.

Then, by the Principle of the Conservation of Vis Viva, we have

$$ma^2 \frac{d\theta^2}{dt^2} + mk^2\omega^2 = 2mga \cos\theta + C.$$

Now, in the case of the solid sphere, $\omega = \frac{d\theta}{dt}$, and therefore

$$(a^2 + k^2) \frac{d\theta^2}{dt^2} = 2ag (\cos\theta - \cos\beta),$$

β being the value of θ when $\frac{d\theta}{dt} = 0$.

In the case of the fluid sphere, by the Principle of the Conservation of Areas, $\omega =$ a constant, and therefore

$$a^2 \frac{d\theta^2}{dt^2} = 2ag (\cos\theta - \cos\beta).$$

Hence, in the former case, the length of the equivalent pendulum is equal to

$$\frac{a^2 + k^2}{a} = a + \frac{2r^2}{5a},$$

and, in the latter, to a.

(7) Two particles P, P', (fig. 231), are connected together by a rigid rod without inertia, which passes through a small smooth ring at O; the rod rests upon a horizontal plane: supposing any impulse whatever to have been communicated to the particles, to find the paths which they will describe.

Let OE be a fixed line in the plane of the motion; let $OP = r$, $PP' = l$; let a, a', be the initial values of OP, OP'; m, m', the masses of P, P'; let $\angle POE = \theta$; let ω, β, be the initial values of $\frac{d\theta}{dt}$, $\frac{dr}{dt}$. Then the differential equation to P's path will be

$$\{mr^2 + m'(l-r)^2\}\{A[mr^2 + m'(l-r)^2] - 1\} = (m + m') \frac{dr^2}{d\theta^2},$$

where $$A = \frac{(m + m')\beta^2 + (ma^2 + m'a'^2)\omega^2}{(ma^2 + m'a'^2)^2\omega^2};$$

and similarly for the path of P'.

Clairaut; *Mém. Acad. Paris*, 1742, p. 38. D'Arcy; *Ib.* 1747, p. 352.

(8) A particle is projected horizontally along the internal surface of a fixed hemisphere the axis of which is vertical and vertex downwards: having given the point of projection, to determine the velocity that the particle may ascend exactly to the rim of the hemisphere.

If a = the radius of the sphere, and β = the inclination to the vertical of the particle's initial distance from the sphere's centre, the required velocity is equal to

$$\left(\frac{2ag}{\cos\beta}\right)^{\frac{1}{2}}.$$

(9) A cone is revolving round its axis with a given angular velocity, when the length of the axis begins to be diminished uniformly, and the vertical angle to be increased so that the volume of the cone remains unchanged: to determine the angular velocity of the cone at the end of any time and the number of revolutions it will make before the motion ceases.

Let ω = the initial angular velocity, h = the initial length, and v = the velocity of decrease of the axis of the cone; then the angular velocity, at the end of a time t, will be equal to

$$\omega\left(1-\frac{vt}{b}\right),$$

and the required number of revolutions is equal to

$$\frac{b\omega}{4\pi v}.$$

(10) One extremity of a string is attached to a ring (supposed to have no weight) which slides along a vertical axis, and the other is attached to a particle of equal mass which moves on a horizontal plane: the particle is projected in a direction perpendicular to the plane which passes through the string and axis: to find the position of the string when it has revolved through a horizontal angle of 90°.

The string will be horizontal, whatever be the initial velocity of the particle or position of the ring.

(11) A screw of Archimedes is capable of turning freely round its axis, which is fixed in a vertical position; a heavy particle is

placed at the top of the tube and runs down through it: to determine the whole angular velocity communicated to the screw.

Let h = the height of the screw, a = the radius of the cylinder, α = the angle which an indefinitely small element of the screw makes with the vertical, ω = the required angular velocity: then, m, m', representing the masses of the screw and particle respectively,

$$\omega^2 = \frac{2m'^2gh}{a^2(m+m') \cdot \{(1+\sin^2\alpha)\,m + \sin^2\alpha \cdot m'\}}.$$

(12) Four equal particles, exercising no attraction on each other, move in an ellipse under the action of a central force in the centre: at the commencement of the motion they were situated at the extremities of the major and minor axes $2a$ and $2b$; if at any time they become suddenly connected with each other so as to form a rigid system, to find the angular velocity of the system about the centre of the ellipse.

If μ = the absolute force in the centre, the system will move about the centre with a constant angular velocity equal to

$$\frac{2ab\mu^{\frac{1}{2}}}{a^2+b^2}.$$

O'Brien and Ellis; *Solutions of the Senate-House Problems for* 1844.

CHAPTER XI.

COEXISTENCE OF SMALL OSCILLATIONS.

CONCEIVE that a particle or a system of particles, subject to certain fixed laws of geometrical connection or constraint, be slightly but generally deranged from a position of stable equilibrium, the invariable elements of the geometry being supposed to be free from particular relations. Then, if in the geometrical equations there be n independent variables, the motion of each member of the system may be represented by the composition of n primary oscillations of different periods, the periods of the n oscillations of any two members of the system being coexistent, while their amplitudes will generally be different. When the periods of the n elementary oscillations are commensurable, the whole system will return to its original state after an interval equal to the least common multiple of these periods; as in the case of vibrating cords and vibrating surfaces. This general property of sympathetic vibrations has been entitled the *Principle of the Coexistence of small Oscillations or Vibrations.*

Should the original derangement of the system from its position of equilibrium, instead of being perfectly general, be effected by peculiar adaptation, we may reduce the n elementary oscillations to any smaller number we may please.

If the fixed geometrical elements of the system be not, as we have supposed, free from particular relations, and if it receive a perfectly general derangement, there will as before arise in the system altogether n classes of oscillations; under these circumstances however a peculiarity occasionally presents itself, viz. that, although as we have supposed the original derangement be quite general, yet into the motion of no single member of the system will all the elementary oscillations enter; this case will then constitute a failure of the Principle of the Coexistence of small Oscillations.

The Principle of Coexistent Oscillations was first laid down by Daniel Bernoulli, who has written several memoirs on the subject in the St. Petersburgh Transactions. See particularly *Nov. Comment. Petrop.* Vol. XIX. p. 281. The student is referred also to Lagrange, *Mécanique Analytique*, Tom. I. p. 347, and to Poisson, *Traité de Mécanique*, Tom. II. p. 426, where he will find investigations of the Principle based on the first principles of Mechanics.

(1) To determine the nature of the oscillations of a material particle within the surface of an ellipsoid, in the neighbourhood of the lower extremity of a vertical axis.

Let $2a$, $2b$, denote the lengths of the two horizontal axes of the ellipsoid, $2c$ representing the length of the vertical one; and let the co-ordinate axes be so chosen that a, b, may be parallel to the axes of x, y, and that c may coincide with the axis of z.

Then, by D'Alembert's Principle combined with the Principle of Virtual Velocities, we have for the motion of the particle,

$$\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\left(\frac{d^2z}{dt^2}+g\right)\delta z=0 \ldots\ldots\ldots\ldots\ldots (1),$$

where x, y, z, denote the co-ordinates of the particle at any time t, and δx, δy, δz, the increments of x, y, z, in passing to any point of the surface very near to the position of the particle.

Again, by the equation to the ellipsoid, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(c-z)^2}{c^2}=1;$$

and therefore, neglecting powers of the small quantities beyond the second,

$$c-z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}=c\left(1-\frac{x^2}{2a^2}-\frac{y^2}{2b^2}\right),$$

$$z=\tfrac{1}{2}c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

$$\delta z=c\left(\frac{x\delta x}{a^2}+\frac{y\delta y}{b^2}\right):$$

hence, from (1), neglecting the products and powers, beyond the first, of small quantities in the coefficients of δx, δy, we get

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{cg}{a^2}\,x\delta x + \frac{cg}{b^2}\,y\delta y = 0,$$

and therefore $$\left(\frac{d^2x}{dt^2} + \frac{cg}{a^2}\,x\right)\delta x + \left(\frac{d^2y}{dt^2} + \frac{cg}{b^2}\,y\right)\delta y = 0.$$

Equating to zero the coefficients of δx, δy, which are independent of each other, we get

$$\frac{d^2x}{dt^2} + \frac{cg}{a^2}\,x = 0 \qquad (2),$$

$$\frac{d^2y}{dt^2} + \frac{cg}{b^2}\,y = 0 \qquad (3).$$

The integral of the equation (2) is

$$x = \beta \sin\left\{\frac{(cg)^{\frac{1}{2}}}{a}\,t + \epsilon\right\},$$

and that of (3) is $$y = \gamma \sin\left\{\frac{(cg)^{\frac{1}{2}}}{b}\,t + \zeta\right\};$$

where $\beta, \gamma, \epsilon, \zeta$, are arbitrary constants, which may be determined from the initial values of $x, y, \frac{dx}{dt}, \frac{dy}{dt}$. It may be observed that the oscillation of the particle depends upon two simple oscillations of which $\frac{\pi a}{(cg)^{\frac{1}{2}}}, \frac{\pi b}{(cg)^{\frac{1}{2}}}$, are the periods; the number of independent simple oscillations being the same as the number of independent variables in the geometrical equation to which the position of the particle is subject.

Poisson: *Traité de Mécanique*, Tom. II. p. 439.

(2) A uniform rod AB, (fig. 232), which is connected by a string OA with a fixed point O, having been slightly displaced from its position of equilibrium; to investigate the nature of its small oscillations.

Draw vertically the indefinite straight line Ox; take P any point in AB, draw PM at right angles to Ox, and produce BA

to meet Ox in C. Let $AB = 2a$, $OM = x$, $PM = y$, $AP = s$, $OA = l$, $\angle AOx = \theta$, $\angle BCx = \phi$.

Then for the motion of the rod we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\int_0^{2a} \left\{ds \left(\frac{d^2x}{dt^2} - g\right) \delta x\right\} + \int_0^{2a} \left\{ds \frac{d^2y}{dt^2} \delta y\right\} = 0 \text{............} (1),$$

where dx, dy, denote the small spaces described by the element ds of the rod in the time dt, parallel to the co-ordinate axes; δx, δy, denoting the resolved parts of its virtual velocity.

Now, from the geometry, we have

$$x = l \cos \theta + s \cos \phi, \quad y = l \sin \theta + s \sin \phi;$$

and therefore, our object being to transform the equation (1) into an equation involving θ, ϕ, instead of x, y, and to retain small quantities only as far as the first order in the coefficients $\delta\theta$, $\delta\phi$, of the new equation, we have approximately

$$x = l(1 - \tfrac{1}{2}\theta^2) + s(1 - \tfrac{1}{2}\phi^2), \qquad y = l\theta + s\phi,$$

$$\delta x = - l\theta\, \delta\theta - s\phi\, \delta\phi, \qquad \delta y = l\delta\theta + s\delta\phi,$$

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = l\frac{d^2\theta}{dt^2} + s\frac{d^2\phi}{dt^2}:$$

hence, substituting these values of x, y,......in the equation (1), we have

$$\int_0^{2a} \{g ds\, (l\theta\, \delta\theta + s\phi\, \delta\phi)\} + \int_0^{2a} \left\{ds \left(l\frac{d^2\theta}{dt^2} + s\frac{d^2\phi}{dt^2}\right)(l\delta\theta + s\delta\phi)\right\} = 0.$$

Equating to zero the coefficient of $\delta\theta$, we get

$$\left(g\theta + l\frac{d^2\theta}{dt^2}\right)\int_0^{2a} ds + \frac{d^2\phi}{dt^2}\int_0^{2a} s\, ds = 0,$$

and therefore $$l\frac{d^2\theta}{dt^2} + a\frac{d^2\phi}{dt^2} + g\theta = 0 \text{....................} (2);$$

and, equating to zero the coefficient of $\delta\phi$, we obtain

$$\left(g\phi + l\frac{d^2\theta}{dt^2}\right)\int_0^{2a} s\, ds + \frac{d^2\phi}{dt^2}\int_0^{2a} s^2\, ds = 0,$$

and therefore $$l\frac{d^2\theta}{dt^2} + \tfrac{4}{3}a\frac{d^2\phi}{dt^2} + g\phi = 0 \text{..................} (3).$$

In order to integrate the equations (2) and (3), assume

$$\theta = \alpha \sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\}, \quad \phi = \beta \sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\};$$

substituting these expressions for θ in (2), and dividing by $\sin\left\{\left(\frac{g}{\rho}\right)^{\frac{1}{2}} t + \epsilon\right\}$, we get

$$-\frac{l\alpha}{\rho} - \frac{a\beta}{\rho} + \alpha = 0, \quad \text{or} \quad a\beta = \alpha(\rho - l) \dots\dots\dots (4);$$

and substituting in (3), we get, in the same way,

$$-\frac{l\alpha}{\rho} - \frac{4a\beta}{3\rho} + \beta = 0, \quad \text{or} \quad \beta(3\rho - 4a) = 3l\alpha \dots\dots\dots (5).$$

Eliminating α and β between the two equations (4) and (5), we obtain

$$\frac{3\rho - 4a}{a} = \frac{3l}{\rho - l}, \quad \text{or} \quad 3\rho^2 - (4a + 3l)\rho + al = 0.$$

Let the two values of ρ deducible from this quadratic be denoted by m, m': then the motion of the rod will be completely determined by the equations

$$\theta = \alpha \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \alpha' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots\dots\dots (6),$$

$$\phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \beta' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots\dots\dots (7).$$

In these two equations there are six arbitrary constants, α, α', β, β', ϵ, ϵ'; they are not however all of them independent of each other; in fact, by (4), since α and α' correspond respectively to the values m and m' of the quantity ρ, we have

$$\beta = \frac{\alpha}{a}(m - l), \quad \beta' = \frac{\alpha'}{a}(m' - l);$$

hence, from (7), we see that

$$\phi = \frac{\alpha}{a}(m - l) \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \frac{\alpha'}{a}(m' - l) \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \dots\dots (8).$$

The four constants $\alpha, \alpha', \epsilon, \epsilon'$, involved in the two equations (7) and (8), may be determined if we have given the initial circumstances of the rod, or the initial values of $\theta, \frac{d\theta}{dt}, \phi, \frac{d\phi}{dt}$.

If $\alpha' = 0$, $\beta' = 0$, then we have

$$\theta = \alpha \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\}, \qquad \phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\},$$

and the oscillations of θ and ϕ will evidently be regular and isochronous, the time of vibration being equal to $\pi\left(\frac{m}{g}\right)^{\frac{1}{2}}$.

If α', β', be not equal to zero, the oscillations of θ and ϕ will be compounded of two simple and isochronous vibrations.

Suppose that at two different times t', t'', the values of θ and of $\frac{d\theta}{dt}$ are the same. This will manifestly be the case if

$$\left(\frac{g}{m}\right)^{\frac{1}{2}} t'' + \epsilon = \left(\frac{g}{m}\right)^{\frac{1}{2}} t' + \epsilon + 2\lambda\pi,$$

and
$$\left(\frac{g}{m'}\right)^{\frac{1}{2}} t'' + \epsilon' = \left(\frac{g}{m'}\right)^{\frac{1}{2}} t' + \epsilon' + 2\lambda'\pi,$$

λ, λ', being any integers; hence

$$2\lambda\pi\left(\frac{m}{g}\right)^{\frac{1}{2}} = t'' - t' = 2\lambda'\pi\left(\frac{m'}{g}\right)^{\frac{1}{2}},$$

and therefore $\qquad \lambda m^{\frac{1}{2}} = \lambda' m'^{\frac{1}{2}}$,

or m, m', must be to each other as two square numbers.

It will be observed that, in agreement with the general theory of the Coexistence of small Oscillations, the number of independent oscillations of θ and ϕ is two, which is the same as the number of the independent geometrical variables.

The following is another method of solving this problem.

Let G (fig. 233) be the position of the centre of gravity of the rod at any time t; draw GH at right angles to the vertical line Ox; let m = the mass of the rod, mk^2 = its moment of inertia about G, T = the tension of the string AO, $OH = x$, $GH = y$. Then, the rest of the notation being the same as before, we have, for the motion of the rod,

$$m\frac{d^2x}{dt^2} = mg - T\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$m\frac{d^2y}{dt^2} = -T\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$mk^2\frac{d^2\phi}{dt^2} = -aT\sin(\phi-\theta) \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Eliminating T between (1) and (2), and omitting small quantities of higher orders than the first, we have

$$\frac{d^2y}{dt^2} + g\theta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

and, eliminating T between (1) and (3), we get in the same manner

$$k^2\frac{d^2\phi}{dt^2} + ag(\phi-\theta) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But $\qquad y = a\sin\phi + l\sin\theta = a\phi + l\theta$, nearly;

hence (4) becomes

$$l\frac{d^2\theta}{dt^2} + a\frac{d^2\phi}{dt^2} + g\theta = 0;$$

and, putting for k^2 its value $\frac{1}{3}a^2$ in (5), we have

$$\tfrac{1}{3}a\frac{d^2\phi}{dt^2} + g(\phi-\theta) = 0.$$

The last two equations are equivalent to the equations (2) and (3) in the former investigation.

Daniel Bernoulli; *Novi Comment. Petrop.* 1773, Tom. XVIII. p. 247. Euler; *Ib.* p. 268.

(3) A pendulum of any figure is firmly attached to a solid circular cylinder as an axis; this axis is supported in a horizontal position at its two extremities, which rest within two hollow circular cylinders placed horizontally and of the same dimensions; to investigate the small oscillations of the pendulum corresponding to any initial state of displacement and motion, the surfaces in contact being considered perfectly smooth.

Let G (fig. 234) be the centre of gravity of the pendulum and its axis, regarded as one mass, at any time of the motion; let the plane of the paper represent the vertical plane through G, which cuts the axis of the solid cylinder at right angles in the point C.

Let the circular arc MAN be the common intersection of the two concave cylinders with the plane of the paper, when produced to meet it. From O, the centre of the arc MAN, draw OAx vertically; GH at right angles to Ox; produce GC to meet Ox in K; join OC, and produce it to a, which will be the point of contact between MAN and the circular section of the solid cylinder made by the plane of the paper. Let $OH=x$, $GH=y$, $AO=a$, $Ca=b$, $\angle AKC=\phi$, $\angle COx=\theta$, $CG=c$; $m=$ the mass of the pendulum and its axis together; $k=$ their radius of gyration about G; $R=$ the reaction of the hollow cylinders against the axis of the pendulum.

Then, for the motion of the pendulum, we have

$$m\frac{d^2x}{dt^2}=mg-R\cos\theta \qquad (1),$$

$$m\frac{d^2y}{dt^2}=-R\sin\theta \qquad (2),$$

$$mk^2\frac{d^2\phi}{dt^2}=-Rc\sin(\phi-\theta) \qquad (3).$$

From (1) and (2) we get, as far as small quantities of the first order,

$$\frac{d^2y}{dt^2}+g\theta=0 \qquad (4);$$

and, from (1) and (3), to the same degree of approximation,

$$k^2\frac{d^2\phi}{dt^2}+cg(\phi-\theta)=0 \qquad (5).$$

Now, from the geometry,

$$y=(a-b)\sin\theta+c\sin\phi$$
$$=(a-b)\theta+c\phi, \quad \text{nearly}:$$

hence from (4) we obtain

$$(a-b)\frac{d^2\theta}{dt^2}+c\frac{d^2\phi}{dt^2}+g\theta=0 \qquad (6).$$

Assume $\theta=\alpha\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t+\epsilon\right\}$, $\quad\phi=\beta\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t+\epsilon\right\}$:

then from (5) we may get

$$\beta\,(cr - k^2) = \alpha cr \text{ .... }(7),$$

and, from (6), $\qquad c\beta = \alpha\,\{r - (a - b)\}$,

and therefore, eliminating α and β,

$$(cr - k^2)\,(r - a + b) = c^2 r.$$

Let the two roots of this quadratic in r be denoted by m and m'; then, for the general values of θ and ϕ, we have

$$\theta = \alpha \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \alpha' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{............ } (8),$$

$$\phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \beta' \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{............ } (9).$$

From (7) we have, β, β', being the values of β, and α, α', those of α, corresponding to the values m, m', of r,

$$\beta = \frac{\alpha cm}{cm - k^2}, \qquad \beta' = \frac{\alpha' cm'}{cm' - k^2};$$

hence, from (9), we have

$$\phi = \frac{\alpha cm}{cm - k^2} \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\} + \frac{\alpha' cm'}{cm' - k^2} \sin\left\{\left(\frac{g}{m'}\right)^{\frac{1}{2}} t + \epsilon'\right\} \text{..... } (10).$$

In the equations (8) and (10) there are four arbitrary constants, α, α', ϵ, ϵ', which may be determined if we have given the initial values of θ, ϕ, $\dfrac{d\theta}{dt}$, $\dfrac{d\phi}{dt}$.

If $\alpha' = 0$, $\beta' = 0$, we have

$$\theta = \alpha \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\}, \quad \phi = \beta \sin\left\{\left(\frac{g}{m}\right)^{\frac{1}{2}} t + \epsilon\right\};$$

and the oscillations of θ and ϕ will be regular and isochronous, the time of vibration being $\pi\left(\dfrac{m}{g}\right)^{\frac{1}{2}}$.

If α' and β' have finite values, the oscillations of θ and ϕ will be compounded of two simple isochronous oscillations.

Euler; *Acta Acad. Petrop.* 1780, P. II. p. 133.

(4) A string $AEFB$ (fig. 235) is attached to two fixed points A, B, in the same horizontal line. From E, F, points so chosen that AE, EF, FB, are all equal, two masses are suspended by strings EM, FN, of different lengths, the masses being equal. Supposing the system to be slightly deranged from its position of equilibrium, to investigate the nature of its small oscillations.

At any time t let EM, FN, make angles ϕ, ϕ', with the vertical. Let AE, EF, BF, make angles $\alpha+\omega$, ω', $\alpha-\omega''$, with the horizon, the values of these angles being α, 0, α, for the position of equilibrium. Draw Mm, Nn, horizontally to meet the vertical line Amn in the points m, n. Let $AE=EF=FB=a$, $EM=k$, $FN=k'$, $Am=x$, $Mm=y$, $An=x'$, $Nn=y'$.

By D'Alembert's Principle and the Principle of Virtual Velocities, we have, for the motion of the system,

$$\left(\frac{d^2x}{dt^2}-g\right)\delta x+\left(\frac{d^2x'}{dt^2}-g\right)\delta x'+\frac{d^2y}{dt^2}\,\delta y+\frac{d^2y'}{dt^2}\,\delta y'=0\ldots\ldots(1).$$

Our object is now to express x, y, x', y', in terms of ω, ϕ, ϕ', and to substitute their values in this equation. This computation must be effected as far as small quantities of the second order.

By the geometry it is plain that

$$a\cos(\alpha+\omega)+a\cos\omega'+a\cos(\alpha-\omega'')=2a\cos\alpha+a,$$

and therefore

$$\cos\alpha\,(1-\tfrac{1}{2}\omega^2)-\sin\alpha.\omega+1-\tfrac{1}{2}\omega'^2+\cos\alpha\,(1-\tfrac{1}{2}\omega''^2)+\sin\alpha.\omega''=2\cos\alpha+1;$$

whence

$$\cos\alpha\,.\,\omega^2+2\sin\alpha\,.\,\omega+\omega'^2+\cos\alpha\,.\,\omega''^2-2\sin\alpha\,.\,\omega''=0\ldots.(2).$$

Again, by the geometry,

$$a\sin(\alpha+\omega)=a\sin\omega'+a\sin(\alpha-\omega''),$$

and therefore

$$\sin\alpha\,(1-\tfrac{1}{2}\omega^2)+\cos\alpha\,.\,\omega=\omega'+\sin\alpha\,(1-\tfrac{1}{2}\omega''^2)-\cos\alpha\,.\,\omega'';$$

whence

$$2\cos\alpha\,.\,\omega-\sin\alpha\,.\,\omega^2=2\omega'-\sin\alpha\,.\,\omega''^2-2\cos\alpha\,.\,\omega''\ldots..(3).$$

Now, as far as the first order of small quantities, we have, from (2),

$$2 \sin \alpha \,.\, \omega = 2 \sin \alpha \,.\, \omega'',$$

and therefore $$\omega'' = \omega\,;$$

and from (3) we have

$$2 \cos \alpha \,.\, \omega = 2\omega' - 2 \cos \alpha \,.\, \omega'' = 2\omega' - 2 \cos \alpha \,.\, \omega,$$

and therefore $$\omega' = 2 \cos \alpha \,.\, \omega.$$

Substituting these values of ω', ω'', in the terms of the second order in (2) and (3), we get

$$(2 \cos \alpha + 4 \cos^2 \alpha)\, \omega^2 + 2 \sin \alpha \,.\, \omega - 2 \sin \alpha \,.\, \omega'' = 0,$$

and $$\cos \alpha \,.\, \omega = \omega' - \cos \alpha \,.\, \omega'' :$$

from these last two equations we see that

$$(2 \cos^2\alpha + 4 \cos^3\alpha)\, \omega^2 + 2 \sin\alpha \cos \alpha . \omega - 2 \sin\alpha . \omega' + 2 \sin\alpha \cos \alpha . \omega = 0,$$

and therefore

$$\omega' = 2 \cos \alpha \,.\, \omega + \frac{\cos^2 \alpha + 2 \cos^3 \alpha}{\sin \alpha}\, \omega^2 \ldots\ldots\ldots\ldots\ldots \;(4).$$

Again, as far as our approximation requires,

$$x = a \sin (\alpha + \omega) + k \cos \phi = a \sin \alpha\,(1 - \tfrac{1}{2}\omega^2) + a \cos \alpha . \omega + k\,(1 - \tfrac{1}{2}\phi^2),$$

$$\delta x = - a \sin \alpha\, \omega \delta\omega + a \cos \alpha\, \delta\omega - k\phi\, \delta\phi,$$

$$\frac{d^2 x}{dt^2} = a \cos \alpha \frac{d^2\omega}{dt^2}\,;$$

$$y = a \cos (\alpha + \omega) + k \sin \phi = a \cos \alpha\,(1 - \tfrac{1}{2}\omega^2) - a \sin \alpha \,.\, \omega + k\phi,$$

$$\delta y = - a \sin \alpha\, \delta\omega + k\, \delta\phi,$$

$$\frac{d^2 y}{dt^2} = - a \sin \alpha \frac{d^2\omega}{dt^2} + k \frac{d^2\phi}{dt^2}\,;$$

$$\begin{aligned} x' &= a \sin (\alpha + \omega) - a \sin \omega' + k' \cos \phi' \\ &= a \sin \alpha\,(1 - \tfrac{1}{2}\omega^2) + a \cos \alpha \,.\, \omega - a\omega' + k'\,(1 - \tfrac{1}{2}\phi'^2) \\ &= a \sin \alpha - a \cos \alpha . \omega - a\, \frac{\sin^2\alpha + 2 \cos^2\alpha + 4 \cos^3\alpha}{2 \sin \alpha}\, \omega^2 + k'\,(1 - \tfrac{1}{2}\phi'^2), \end{aligned}$$

by (4);

$$\delta x' = - a \cos \alpha\, \delta\omega - a\, \frac{\sin^2 \alpha + 2 \cos^2 \alpha + 4 \cos^3 \alpha}{\sin \alpha}\, \omega\, \delta\omega - k'\phi' \delta\phi',$$

$$\frac{d^2 x'}{dt^2} = - a \cos \alpha \frac{d^2\omega}{dt^2}\,;$$

$$y' = a\cos(\alpha+\omega) + a\cos\omega' + k'\sin\phi'$$
$$= a\cos\alpha\,(1-\tfrac{1}{2}\omega^2) - a\sin\alpha\,.\,\omega + a\,(1-\tfrac{1}{2}\omega'^2) + k'\phi',$$
$$\delta y' = k'\delta\phi' - a\sin\alpha\,\delta\omega,$$
$$\frac{d^2y'}{dt^2} = k'\frac{d^2\phi'}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2}.$$

Hence, by the equation (1), there is

$$\left.\begin{array}{l} 2a^2\cos^2\alpha\dfrac{d^2\omega}{dt^2}\delta\omega + kg\phi\delta\phi + ga\dfrac{2+4\cos^2\alpha}{\sin\alpha}\omega\delta\omega \\ \quad + gk'\phi'\delta\phi' + \left(a\sin\alpha\dfrac{d^2\omega}{dt^2} - k\dfrac{d^2\phi}{dt^2}\right)(a\sin\alpha\,\delta\omega - k\delta\phi) \\ \qquad + \left(k'\dfrac{d^2\phi'}{dt^2} - a\sin\alpha\dfrac{d^2\omega}{dt^2}\right)(k'\delta\phi' - a\sin\alpha\,\delta\omega) \end{array}\right\} = 0.$$

Hence, equating to zero the coefficients of the independent quantities $\delta\phi$, $\delta\phi'$, $\delta\omega$, we get

$$k\frac{d^2\phi}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2} + g\phi = 0 \ldots\ldots\ldots\ldots\ (5),$$

$$k'\frac{d^2\phi'}{dt^2} - a\sin\alpha\frac{d^2\omega}{dt^2} + g\phi' = 0 \ldots\ldots\ldots\ldots\ (6),$$

$$2a\frac{d^2\omega}{dt^2} - k\sin\alpha\frac{d^2\phi}{dt^2} - k'\sin\alpha\frac{d^2\phi'}{dt^2} + g\frac{2+4\cos^2\alpha}{\sin\alpha}\omega = 0 \ldots.. (7).$$

Eliminating $\dfrac{d^2\phi}{dt^2}$ and $\dfrac{d^2\phi'}{dt^2}$ between (5), (6), (7), we get

$$2a\sin\alpha\cos^2\alpha\frac{d^2\omega}{dt^2} + g\{(2+4\cos^2\alpha)\,\omega + \sin^2\alpha\,.\,\phi + \sin^2\alpha\,.\,\phi'\} = 0 \ldots.. (8).$$

Let r denote the length of a pendulum, isochronous with one of the elementary oscillations, and assume accordingly

$$\omega = \Omega\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\},$$

$$\phi = F\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\},$$

$$\phi' = F'\sin\left\{\left(\frac{g}{r}\right)^{\frac{1}{2}}t + \epsilon\right\}.$$

Then, from (5), (6), (8), we have

$$(k-r)\,F = a\sin\alpha\,.\,\Omega,$$
$$(k'-r)\,F' = a\sin\alpha\,.\,\Omega,$$

$$-2a\sin\alpha\cos^2\alpha\,\frac{1}{r}\Omega+(2+4\cos^2\alpha)\Omega+\sin^2\alpha.F+\sin^2\alpha.F'=0\,;$$

and, by eliminating the constants F, F', Ω, we have a cubic equation in r,

$$\frac{2\sin\alpha\cos^2\alpha}{r}+\frac{\sin^3\alpha}{r-k}+\frac{\sin^3\alpha}{r-k'}-\frac{2+4\cos^2\alpha}{a}=0\,\ldots\ldots\ldots(9).$$

Let l, l', l'', be the three roots of this equation; then, for the complete solution of the problem, we have

$$\omega''=\omega=\Omega\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+\Omega'\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+\Omega''\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\},$$

$$\omega'=2\cos\alpha.\omega,$$

$$\phi=F_1\sin\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+F_2\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+F_3\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\},$$

$$\phi'=F_1'\sin\left\{\left(\frac{g}{l}\right)^{\frac{1}{2}}t+\epsilon\right\}+F_2'\sin\left\{\left(\frac{g}{l'}\right)^{\frac{1}{2}}t+\epsilon'\right\}+F_3'\sin\left\{\left(\frac{g}{l''}\right)^{\frac{1}{2}}t+\epsilon''\right\}.$$

This problem may be solved also in the following manner, which is Euler's method of considering it.

Let P, Q, be the tensions of the strings EM, FN, and m the mass of each of the bodies.

Then, for the motion of the bodies, we have, approximately,

$$m\frac{d^2x}{dt^2}=mg-P\cos\phi=mg-P\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$m\frac{d^2y}{dt^2}=-P\sin\phi=-mg\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$m\frac{d^2x'}{dt^2}=mg-Q\cos\phi'=mg-Q\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$m\frac{d^2y'}{dt^2}=-Q\sin\phi'=-mg\phi'\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)\,;$$

these four equations being true as far as the first order of small quantities.

Let T denote the tension of the string EF; then, since the three tensions acting upon the point E must be in equilibrium, there is

$$\frac{T}{P}=\frac{\sin\{\phi+\frac{1}{2}\pi+\alpha+\omega\}}{\sin\{\frac{1}{2}\pi-(\alpha+\omega)+\frac{1}{2}\pi-\omega'\}}=\frac{\cos(\alpha+\omega+\phi)}{\sin(\alpha+\omega+\omega')}.$$

Similarly, for the tensions at F, we have

$$\frac{Q}{T}=\frac{\sin(\alpha-\omega''-\omega')}{\cos(\alpha-\omega''-\phi')};$$

hence
$$\frac{Q}{P}=\frac{\sin(\alpha-\omega''-\omega')\cos(\alpha+\omega+\phi)}{\cos(\alpha-\omega''-\phi')\sin(\alpha+\omega+\omega')}.$$

Hence, as far as small quantities of the first order,

$$Q\{\sin\alpha+\cos\alpha(\omega+\omega')\}\{\cos\alpha+\sin\alpha(\omega''+\phi')\}$$
$$=P\{\sin\alpha-\cos\alpha(\omega'+\omega'')\}\{\cos\alpha-\sin\alpha(\omega+\phi)\},$$

and therefore

$$\left.\begin{aligned}&Q\{\sin\alpha\cos\alpha+\sin^2\alpha(\omega''+\phi')+\cos^2\alpha(\omega+\omega')\}\\&=P\{\sin\alpha\cos\alpha-\sin^2\alpha(\omega+\phi)-\cos^2\alpha(\omega'+\omega'')\}\end{aligned}\right\}\text{........(5).}$$

Now, by the geometry,

$$\cos(\alpha+\omega)+\cos\omega'+\cos(\alpha-\omega'')=2\cos\alpha+1,$$

and therefore, as far as the first order of small quantities,

$$-\sin\alpha\,.\,\omega+\sin\alpha\,.\,\omega''=0,$$
$$\omega''=\omega\text{..............................(6).}$$

Also, by the geometry,

$$\sin(\alpha+\omega)=\sin\omega'+\sin(\alpha-\omega''),$$
$$\cos\alpha\,.\,\omega=\omega'-\omega''\cos\alpha=\omega'-\omega\cos\alpha,$$
$$\omega'=2\omega\cos\alpha\text{..........................(7).}$$

Hence by (5), (6), (7), we have

$$\left.\begin{aligned}&Q\{\sin\alpha\cos\alpha+\sin^2\alpha(\omega+\phi')+\cos^2\alpha(1+2\cos\alpha)\,\omega\}\\&=P\{\sin\alpha\cos\alpha-\sin^2\alpha(\omega+\phi)-\cos^2\alpha(1+2\cos\alpha)\,\omega\}\end{aligned}\right.\text{.....(8).}$$

Eliminating P and Q between (1), (3), and (8), we have, as far as small quantities of the first order,

$$\left(\frac{d^2x}{dt^2}-\frac{d^2x'}{dt^2}\right)\sin\alpha\cos\alpha=-g\{(2+4\cos^2\alpha)\omega+\sin^2\alpha.\phi+\sin^2\alpha.\phi'\}..(9).$$

But $x=a\sin(\alpha+\omega)+k\cos\phi=a\cos\alpha\,.\,\omega+\ldots,$

$$y=a\cos(\alpha+\omega)+k\sin\phi=-a\sin\alpha\,.\,\omega+k\phi+\ldots,$$

$$x' = a\sin(\alpha+\omega) - a\sin\omega' + k'\cos\phi' = a\cos\alpha\,.\,\omega - a\omega' + \ldots$$
$$= -a\cos\alpha\,.\,\omega + \ldots,$$
$$y' = a\cos(\alpha+\omega) + a\cos\omega' + k'\sin\phi' = -a\sin\alpha\,.\,\omega + k'\phi' + \ldots;$$

hence, by (9),

$$2a\sin\alpha\cos^2\alpha\,\frac{d^2\omega}{dt^2} + g\,\{(2+4\cos^2\alpha)\,\omega + \phi\sin^2\alpha + \phi'\sin^2\alpha\} = 0:$$

and, by (2) and (4),

$$k\,\frac{d^2\phi}{dt^2} - a\sin\alpha\,\frac{d^2\omega}{dt^2} + g\phi = 0,$$

$$k'\,\frac{d^2\phi'}{dt^2} - a\sin\alpha\,\frac{d^2\omega}{dt^2} + g\phi' = 0;$$

which are the same three linear equations as (5), (6), (8), in the former solution.

If k be equal to k', the cubic equation (9) of the former solution will degenerate into a quadratic, and the variations of ω, ϕ, ϕ', will no longer be expressible by the composition of the same elementary vibrations. This will be an instance of the failure of the Principle of the Coexistence of small Oscillations.

Euler; *Act. Acad. Petrop.* 1779, P. II. p. 95.

(5) A heavy hollow circular ring is suspended by a point in its circumference, and a heavy particle is placed inside it: they are both made to oscillate through a small extent from their positions of equilibrium, in the plane of the ring; to determine the number and periods of the coexistent oscillations of the system.

If a denote the radius of the ring, and M, m, the masses of the ring and particle respectively, there will be in the system two coexistent oscillations the periods of which are

$$\pi\left(\frac{2a}{g}\right)^{\frac{1}{2}} \text{ and } \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}.\left(\frac{M}{M+m}\right)^{\frac{1}{2}}.$$

(6) A thin hemispherical bowl rocks on a horizontal plane sufficiently rough to prevent sliding, and has attached to its centre a fine string, of half the length of the radius, with a particle of equal mass with itself at the free extremity; to determine

the number and the periods of the small oscillations of the system, supposing the motion to be such that all the molecules of the system move parallel to one vertical plane.

There will be two coexistent oscillations, the periods of which are equal to the two values of $\frac{\pi}{\sqrt{\rho}}$; ρ being given by the equation

$$\rho^2 - \frac{23}{4} \cdot \frac{g}{r} \cdot \rho + \frac{3}{2} \cdot \frac{g^2}{r^2} = 0,$$

r denoting the radius of the bowl.

(7) One of the scales of a common balance having been slightly displaced from its position of rest, in a vertical plane passing through the beam; to investigate the nature of the oscillatory motions of the two scales and of the beam, to which the displacement will give rise.

Let O (fig. 236) be the point of suspension of the whole balance, G its centre of gravity, AB the beam, P and Q the scales, which are here supposed to be material points. Draw aOb horizontal, $aA\alpha$, $bB\beta$, vertical. Let $AC = a = BC$, $OC = b$, $OG = c$, $AP = l = BP$, $Mk^2 =$ the moment of inertia of the beam about O, $m =$ the mass of P and of Q supposed to be equal.

Let ϕ be the angle which, at any time t, the beam makes with the horizon; let $\angle PA\alpha = \eta$, $\angle QB\beta = \theta$. Also put

$$\frac{g}{l} = n^2, \quad \frac{b}{l} = h, \quad \frac{Mc + 2mb}{Mk^2} g = p^2, \quad -\frac{mbg}{Mk^2} = q,$$

and let $-\mu_1^2$, $-\mu_2^2$, represent the two roots of the quadratic

$$z^2 + (n^2 + p^2 - 2hq)\, z + n^2p^2 = 0.$$

Then, bearing in mind that, initially,

$$\phi = 0, \qquad \eta = e, \qquad \theta = 0,$$

$$\frac{d\phi}{dt} = 0, \quad \frac{d\eta}{dt} = 0, \quad \frac{d\theta}{dt} = 0,$$

where ϵ is a known constant, we shall obtain for the complete expression of the motions

$$\phi = \frac{2\epsilon h p^2 q^2}{\mu_2^2 - \mu_1^2} (\cos \mu_1 t - \cos \mu_2 t),$$

$$2\eta = \frac{2\epsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2}\right) + e \cos nt,$$

$$2\theta = \frac{2\epsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2}\right) - e \cos nt.$$

(8) One of the scales of a common balance having been slightly displaced from its position of rest, in a vertical plane at right angles to the beam; to investigate the nature of the oscillatory motions of the two scales and of the beam.

Let AB (fig. 237) be the original position of the beam, PQ its position at any time t; p, q, the projections of the positions of the scales considered as material points at the same time. Let $AC = a = BC$, $AP = z = BQ$, $Pp = x$, $Qq = y$, $Mk^2 =$ the moment of inertia of the beam round C, $m =$ the mass of each scale, $l =$ the length of the string by which each scale is suspended. If we put, for simplicity,

$$\frac{g}{l} = n^2, \quad \frac{g}{l}\left(1 + 2\frac{ma^2}{Mk^2}\right) = n'^2,$$

we shall have, for the complete expression of the motions, the initial value of x being c, while those of y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, are all zero;

$$x = c \cos\left(\frac{n' - n}{2}\, t\right) \cos\left(\frac{n' + n}{2}\, t\right),$$

$$y = -c \sin\left(\frac{n' - n}{2}\, t\right) \sin\left(\frac{n' + n}{2}\, t\right),$$

$$z = \frac{2g}{l} \frac{ma^2}{Mk^2} \frac{c}{n'^2} \sin^2 \frac{n't}{2}.$$

Investigations of the two last problems are given in a paper on the Sympathy of Pendulums, in the *Cambridge Mathematical Journal*, Vol. II. p. 120.

CHAPTER XII.

IMPULSIVE FORCES.

If two rigid bodies impinge against each other, their motions both of translation and of rotation will generally experience modification, the determination of the nature of which, in the case of bodies of which the positions and motions are assigned at the instant before impact, constitutes the general problem of collision. The process of collision may be divided into two stages of indefinitely small duration: in the former stage, by the force of compression, which we will denote by R, the two points in which the bodies touch each other are constrained to assume equal resolved velocities in the direction of the common normal to their surfaces; in the latter stage, by the force of restitution, if the bodies be not inelastic, an additional reaction eR takes place between them, where e denotes their common elasticity. Let ω_1, ω_2, ω_3, denote the angular velocities of one of the bodies about its principal axes and v_1, v_2, v_3, the components of the velocity of its centre of gravity, at the conclusion of the former stage of the collision; let ω_1', ω_2', ω_3', v_1', v_2', v_3', denote the analogous quantities in relation to the other body. Then, for the expression of the motion of the former body, as modified by the force of compression, we shall have six equations involving, together with known quantities, the symbols ω_1, ω_2, ω_3, v_1, v_2, v_3, R; and in like manner for the latter body we shall have six equations involving ω_1', ω_2', ω_3', v_1', v_2', v_3', R. Thus we shall have in all twelve equations involving thirteen variables. Another equation is supplied by the condition that the points of the two bodies in which their contact takes place shall have an equal resolved velocity in the direction of the common normal. Thus we shall be able to determine completely the modification of the motions of the two bodies due to the force of compression as well as the magnitude

of this force. An additional modification must be applied, in the case of elastic bodies, in consequence of the force of restitution eR, which, from the investigation for the former stage of the collision, has become a known force. If one of the bodies be immovable, the simplification of the method of investigation which we have described is obvious, the thirteen equations of which we made mention being reduced in this case to seven, and the common normal velocity of the two points of contact being zero. For ample information on this subject the student is referred to Poisson's *Traité de Mécanique*, Tom. II. p. 254, seconde édition.

SECT. 1. *Single Body. Smooth Surfaces.*

(1) A beam of imperfect elasticity, moving anyhow in a vertical plane, impinges upon a smooth horizontal plane; to determine the initial motion of the beam after impact.

We will commence with supposing the beam to be inelastic; in this case the extremity of the beam which strikes the horizontal plane will continue after impact to slide along it without detaching itself. Let PQ (fig. 238) represent the beam at any time after collision; KL being the section of the horizontal plane made by the vertical plane through PQ; G the centre of gravity of PQ; draw GH at right angles to KL. Let $GH=y$, $QG=a$, $\angle GQH=\theta$, $k=$ the radius of gyration about G, $m=$ the mass of the beam; ω, ω', the angular velocities of the beam about G estimated in the direction of the arrows in the figure, just before and just after impact; u, v, the vertical velocities of G estimated downwards just before and just after impact; B the blow of collision.

Then, $\omega'-\omega$ being the angular velocity communicated by the blow, we shall have, if β be the value of θ at the instant of impact,

$$\omega'-\omega=\frac{Ba\cos\beta}{mk^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and, $u-v$ being the velocity of G which is destroyed by the blow,

$$u-v=\frac{B}{m}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, by the geometry, we get

$$y = a \sin\theta;$$

and therefore, t denoting the interval between the instant of collision and the arrival of the beam at the position represented in the figure,

$$\frac{dy}{dt} = a \cos\theta \frac{d\theta}{dt};$$

hence, $-v$, $-\omega'$, being the values of $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, at the instant after collision, we have

$$v = a \cos\beta . \omega' \text{.........................} (3).$$

From (1), (2), (3), we get

$$u - \frac{B}{m} = a \cos\beta \left(\omega + \frac{Ba \cos\beta}{mk^2}\right),$$

$$\frac{B}{m}\left(1 + \frac{a^2}{k^2}\cos^2\beta\right) = u - a\omega \cos\beta,$$

$$B = mk^2 \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2} \text{....................} (4).$$

Hence, from (1),

$$\omega' = \omega + a \cos\beta \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2} = \frac{au \cos\beta + k^2\omega}{a^2 \cos^2\beta + k^2};$$

and, from (2),

$$v = u - k^2 \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2} = a \cos\beta \frac{au \cos\beta + k^2\omega}{a^2 \cos^2\beta + k^2}.$$

Next let us suppose the beam to be imperfectly elastic, its elasticity being denoted by e; in this case the value of B given in (4) must be increased in the ratio of $1 + e$ to 1; and therefore, instead of the equation (4), we have

$$B = (1 + e)\, mk^2 \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2},$$

which determines the magnitude of the blow of impact: substituting this value of B in (1), we get

$$\omega' = \omega + (1 + e)\, a \cos\beta \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2}$$

$$= \frac{(k^2 - a^2 e \cos^2\beta)\, \omega + (1 + e)\, au \cos\beta}{a^2 \cos^2\beta + k^2};$$

and, substituting in (2),

$$v = u - (1+e)\,k^2 \frac{u - a\omega \cos\beta}{a^2 \cos^2\beta + k^2}$$

$$= \frac{a^2 u \cos^2\beta - ek^2 u + (1+e)\,k^2 a\omega \cos\beta}{a^2 \cos^2\beta + k^2}.$$

The velocity of G parallel to the plane KL will be the same before and after impact. The end B of the beam will evidently after collision detach itself from the horizontal plane, since v is less and ω' greater when e has a finite value than when it is equal to zero.

(2) A body AB, (fig. 239), after sliding from a given altitude down an inclined plane Oy, impinges against a small obstacle at C; to determine the impulsive reaction of the obstacle and the motion of the body immediately after collision.

Let G be the centre of gravity of the body; draw GH at right angles to Oy; Ox parallel to HG. Let $GH = a$, $CH = b$, $m =$ the mass of the body, $k =$ the radius of gyration about G; $c =$ the velocity of G immediately before impact. We will commence with supposing the body to be perfectly inelastic; in this case the point C of the body will remain during collision in contact with the obstacle, the body rotating about this point. Let R, S, denote the impulsive reactions of the obstacle parallel to Ox, yO; and let u, v, denote the velocities of G parallel to Ox, Oy, on the completion of the impact; also let ω represent the angular velocity of rotation about G at the same instant.

Then we have, for the motion of translation,

$$mu = R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$mv = mc - S \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and, for the motion of rotation,

$$mk^2\omega = Sa - Rb \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, the velocity of the point C of the body, estimated parallel to Ox, will be

$$u - \omega CG \cos \angle GCH \text{ or } u - b\omega,$$

the former term of this expression arising from the motion of G, and the latter from the rotation of the body about G.

Also the velocity of the point C, parallel to Oy, will be

$$v - \omega . CG \sin \angle GCH \text{ or } v - a\omega,$$

the former term being due to the motion of G, and the latter to the rotation about G. But the point C of the body, which is perfectly inelastic, remains at rest during the collision; hence, evidently,

$$u - b\omega = 0 \ldots\ldots(4); \qquad v - a\omega = 0 \ldots\ldots\ldots\ldots(5).$$

From (1) and (4) we have

$$R = mb\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and from (2), (5),

$$S = m(c - a\omega) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

substituting these values of R and S in (3), we obtain

$$k^2\omega = ac - a^2\omega - b^2\omega,$$

and therefore,

$$\omega = \frac{ac}{a^2 + b^2 + k^2}, \quad u = \frac{abc}{a^2 + b^2 + k^2}, \quad v = \frac{a^2c}{a^2 + b^2 + k^2};$$

hence also, from (6),

$$R = \frac{mabc}{a^2 + b^2 + k^2};$$

and, from (7),

$$S = m\left(c - \frac{a^2c}{a^2 + b^2 + k^2}\right) = \frac{mc(b^2 + k^2)}{a^2 + b^2 + k^2}.$$

If the body be supposed to be elastic, we must increase these values of R and S in the ratio of $1 + e$ to 1, e denoting the elasticity. Hence

$$R = \frac{m(1 + e)abc}{a^2 + b^2 + k^2}, \quad S = \frac{mc(1 + e)(b^2 + k^2)}{a^2 + b^2 + k^2};$$

and therefore, from (1), (2), (3),

$$u = \frac{(1 + e)abc}{a^2 + b^2 + k^2}, \quad v = c\frac{a^2 - e(b^2 + k^2)}{a^2 + b^2 + k^2}, \quad \omega = \frac{(1 + e)ac}{a^2 + b^2 + k^2}.$$

(3) A beam AB (fig. 240) is originally in a vertical position, hanging from the point O along the line Oy; supposing the extremity A of the beam to be projected from O with a given velocity along a smooth horizontal groove Ox, to determine the motion of the beam.

Let AB be the position of the beam after a time t from the projection of A, G its centre of gravity; draw GH at right angles to Ox: let $OH = x$, $GH = y$, $\angle OAG = \theta$, $AG = a$; $m =$ the mass of the beam, $k =$ its radius of gyration about G.

Then, for the motion of the beam at any time after the projection, we have, by the Principle of the Conservation of Vis Viva,

$$m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C + 2mgy \ldots\ldots\ldots\ldots(1);$$

and, by the Principle of the Conservation of the Motion of the Centre of Gravity,

$$\frac{dx}{dt} = C' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), we have

$$m\left(\frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right) = C'' + 2mgy;$$

but, from the geometry, we see that $y = a\sin\theta$; hence

$$m(a^2\cos^2\theta + k^2)\frac{d\theta^2}{dt^2} = C'' + 2mga\sin\theta \ldots\ldots\ldots\ldots(3).$$

Let B denote the blow of projection which is impressed upon the end A of the beam; u the velocity of A's projection, and ω the angular velocity of the beam about G immediately after the blow. Also let v be the velocity communicated to G by the blow.

Then we shall have

$$mv = B, \qquad mk^2\omega = Ba, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, the velocity of A along Ox will be equal to

$$v + a\omega,$$

the former term being due to the motion of G, and the latter to

the rotation about G; but the velocity of A is also u by the hypothesis; hence

$$v + a\omega = u;$$

but, from the equations (4), we have $k^2\omega = av$; we obtain, therefore,

$$u = v + \frac{a^2}{k^2}v, \quad v = \frac{k^2 u}{a^2 + k^2}, \quad \omega = \frac{au}{a^2 + k^2}.$$

Now, $\theta = \frac{1}{2}\pi$, $\frac{d\theta}{dt} = \omega$, simultaneously; hence, from (3),

$$mk^2\omega^2 = C'' + 2mga;$$

and therefore

$$(a^2\cos^2\theta + k^2)\frac{d\theta^2}{dt^2} = k^2\omega^2 - 2ga(1 - \sin\theta)$$

$$= \frac{k^2 a^2 u^2}{(a^2 + k^2)^2} - 2ga(1 - \sin\theta) \ldots\ldots(5).$$

Also, the value of $\frac{dx}{dt}$ being constant, as is shewn by the equation (2),

$$\frac{dx}{dt} = v = \frac{k^2 u}{a^2 + k^2}, \quad x = \frac{k^2 ut}{a^2 + k^2};$$

which gives the velocity of G parallel to Ox, and the value of x at any time of the motion; the angular velocity of the beam for every position is given by (5), whence θ is to be ascertained in terms of t.

(4) An inelastic beam AB, (fig. 241), capable of moving in a vertical plane about a fixed horizontal axis through A, falls from a given position, and impinges against an immoveable obstacle at C; to determine the shock on the axis.

Let G be the centre of gravity of the beam; AM a horizontal line through A; let m = the mass of the beam; $\angle GAM = \theta$ at any time t of the descent; α = the initial value of θ; k = the radius of gyration about G; $AG = a$.

Then, for the motion of the beam in its fall,

$$m(a^2 + k^2)\frac{d^2\theta}{dt^2} = mga\cos\theta;$$

multiplying by $2\frac{d\theta}{dt}$ and integrating

$$m(a^2+k^2)\frac{d\theta^2}{dt^2} = 2mag\sin\theta + C;$$

but $\theta = \alpha$ when $\frac{d\theta}{dt} = 0$; hence

$$0 = 2mag\sin\alpha + C,$$

and therefore $\quad (a^2+k^2)\frac{d\theta^2}{dt^2} = 2ag(\sin\theta - \sin\alpha).$

Let $\angle\, CAM = \beta$ and, at the instant before collision, let $\frac{d\theta}{dt} = \omega$; then, clearly,

$$(a^2+k^2)\,\omega^2 = 2ag(\sin\beta - \sin\alpha)\ldots\ldots\ldots\ldots\ldots(1).$$

Let R, R', denote the impulsive reactions of the obstacle C and the axis A, at the instant of impact; both of which will evidently be at right angles to the length of the beam. Now the effect of the reaction R is to destroy the whole of the angular velocity of the beam about A, by impressing upon it an equal and opposite angular velocity; hence, putting $CA = c$,

$$m\omega(a^2+k^2) = Rc\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, the difference of the moments of R and R' about the centre of gravity of the beam being

$$R(c-a) - R'a,$$

we must have

$$R(c-a) - R'a = mk^2\omega\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3) we obtain

$$m\omega(a^2+k^2)(c-a) - R'ac = mk^2c\omega,$$

$$R'ac = m\omega\{(c-a)(a^2+k^2) - ck^2\},$$

$$R' = m\omega\left\{a - \frac{a^2+k^2}{c}\right\};$$

and therefore, from (1),

$$R' = m(2ag)^{\frac{1}{2}}\left(\frac{\sin\beta - \sin\alpha}{a^2+k^2}\right)^{\frac{1}{2}}\left(a - \frac{a^2+k^2}{c}\right).$$

If $R' = 0$, we must have

$$a - \frac{a^2 + k^2}{c} = 0, \quad c = \frac{a^2 + k^2}{a};$$

and therefore C must be the centre of oscillation of the beam at the moment of collision.

If the beam be elastic, we must increase the value of R given by (2) in the ratio of $1 + e$ to 1, e denoting the elasticity; we shall then have, from (3),

$$R' = \frac{m\omega}{ac}\{(1 + e)(c - a)(a^2 + k^2) - ck^2\}.$$

(5) An inelastic beam, which is moving without rotation along a smooth horizontal plane, impinges upon a fixed rod at right angles to the plane; to determine the impulsive reaction of the rod and the motion of the beam subsequent to the impact.

Let AB (fig. 242) be the position of the beam at the instant of impact; O the place of the obstacle; G the centre of gravity of the beam; $G'G$ the line of G's motion before collision. Produce OB indefinitely to x, and draw the indefinite line yOy' at right angles to Ox and meeting GG' in G'. Let $R =$ the impulsive reaction of O, which will be exerted along the line Oy'; $u =$ the velocity of G before impact; $\angle OG'G = \alpha$; $OG = c$; $k =$ the radius of gyration of AB about G; $m =$ the mass of the beam; let v_x, v_y, be the velocities of G parallel to Ox, Oy, just after impact, and ω the angular velocity about G.

Then, by the equations of impulsive motion,

$$mv_x = mu \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$mv_y = mu \cos \alpha - R \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$mk^2\omega = Rc \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, the velocity of the point O of the beam in the direction Oy, the instant after impact, must be $v_y - c\omega$, v_y being its velocity due to the velocity of G, and $-c\omega$ its velocity due to the rotation of the beam about G; but, the beam being inelastic, the effect of the impact is to destroy the resolved part of O's velocity at right angles to AB; hence v_y must be equal to $c\omega$.

We have, then, from (2),

$$mc\omega = mu \cos\alpha - R,$$

and therefore, by the aid of (3),

$$mc^2\omega = mcu \cos\alpha - mk^2\omega,$$

or

$$\omega = \frac{cu \cos\alpha}{c^2 + k^2}.$$

Hence, from (3),

$$R = \frac{mk^2u \cos\alpha}{c^2 + k^2};$$

and consequently, from (2),

$$v_y = u \cos\alpha - \frac{k^2u \cos\alpha}{c^2 + k^2} = \frac{c^2u \cos\alpha}{c^2 + k^2}.$$

Also, from (1), $v_x = u \sin\alpha.$

Thus we have determined completely the instantaneous motions of the beam after the impact, and the impulsive reaction of the rod at O.

It may be ascertained that, if the original motion be precisely such as our particular figure represents it, on the consummation of the impact, the beam will detach itself from the obstacle and will then move along freely with the velocities v_x, v_y, ω, which we have obtained above. In fact we should find, if we were to assume the beam always to touch the obstacle, that the obstacle would have to exert a continuous attraction instead of a reaction.

(6) An inelastic cylinder rolls without sliding along a plane, and impinges upon a perfectly rough point, the circular section of the cylinder through the rough point being supposed to bisect the axis of the cylinder: to determine the least distance of the point from the plane in order that the cylinder may be reduced to rest by the impact.

Let ω be the angular velocity of the cylinder before and ω' just after impact; the cylinder being supposed to turn over the fixed point C, (fig. 243). Then, a being the radius of the cylinder, the centre O of a transverse section will have a horizontal velocity $a\omega$ before impact, and a velocity $a\omega'$, at right

angles to the radius CO, just after impact. Let c denote the distance of C from the plane on which the cylinder is rolling, R the normal and S the tangential reaction at C.

Then, for the motion of translation at right angles to CO, we have

$$ma\omega' = ma\omega \cdot \frac{a-c}{a} + S \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and, for rotation about O, there is

$$\tfrac{1}{2}ma^2\omega' = \tfrac{1}{2}ma^2\omega - Sa \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$\omega' = \omega \cdot \frac{3a-2c}{3a}.$$

This result shews that the cylinder will roll over the fixed point if c be less than $\frac{3}{2}a$.

The following is a different solution of the same problem.

The motion the instant before impact is made up of two motions, the one of translation, the velocity being $a\omega$, and the other of rotation, the angular velocity being ω.

Let P be any point in the area of the circular section through C; let $OP = r$, and let $\theta =$ the inclination of OP to CO, and ϕ its inclination to the horizontal line AB. Then the moment of the momentum of the circular section about C, due to the rotation, is equal to

$$\int_0^a \int_0^{2\pi} \rho r\, d\theta\, dr \,.\, r\omega \,.\, (r + a\cos\theta)$$

$$= \rho\omega \int_0^a \int_0^{2\pi} r^2 d\theta\, dr\, (r + a\cos\theta)$$

$$= \rho\omega a^4 \int_0^{2\pi} d\theta\, (\tfrac{1}{4} + \tfrac{1}{3}\cos\theta) = \tfrac{1}{2}\pi\rho\omega a^4.$$

The moment of the momentum, due to translation, is equal to

$$\int_0^a \int_0^{2\pi} a\omega\rho r d\phi dr\, (r\sin\phi + a - c)$$

$$= a^2\omega\rho \int_0^{2\pi} d\phi\, \{\tfrac{1}{3}a\sin\phi + \tfrac{1}{2}(a-c)\}$$

$$= \pi\rho\omega a^3 (a-c).$$

Hence the whole moment is equal to

$$\tfrac{1}{3}\pi\rho\omega a^3 (3a - 2c),$$

which will not be positive unless c be less than $\frac{3}{2}a$, that is, if the cylinder be reduced to rest, c will be not less than $\frac{3}{2}a$.

(7) A rigid system at rest is struck by any system of simultaneous blows: to determine the position and velocity of the Spontaneous Axis of Rotation, that is, of a straight line, rigidly connected with the system, which, on the application of the blows, has no motion but in the direction of its length.

Let the centre of gravity of the system be taken as the origin of co-ordinates: the system of blows may be reduced to three impulsive pressures X, Y, Z, at the origin, along the axes of x, y, z, respectively, and three impulsive couples the moments of which are L, M, N, in the planes yz, zx, xy, respectively.

Let V_x, V_y, V_z, be the components of the absolute velocity of a particle δm, (the co-ordinates of which are x, y, z), just after the impacts, parallel to the axes of co-ordinates; V_x', V_y', V_z', the components of the velocity of the same particle relatively to the centre of gravity; $\overline{V}_x$, $\overline{V}_y$, $\overline{V}_z$, the components of the velocity of the centre of gravity; ω_1, ω_2, ω_3, the angular velocities impressed upon the system about the three axes.

Then we have

$$V_x = \overline{V}_x + V_x', \qquad V_y = \overline{V}_y + V_y', \qquad V_z = \overline{V}_z + V_z';$$

and

$$\left.\begin{aligned} V_x' &= z\omega_2 - y\omega_3 \\ V_y' &= x\omega_3 - z\omega_1 \\ V_z' &= y\omega_1 - x\omega_2 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now we have, for the motion about the centre of gravity, the equations

$$\left.\begin{aligned} \Sigma\delta m\,(yV_z' - zV_y') &= L \\ \Sigma\delta m\,(zV_x' - xV_z') &= M \\ \Sigma\delta m\,(xV_y' - yV_x') &= N \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(2);$$

which, by substituting for V_x', V_y', V_z', the values given above, become

$$\left.\begin{aligned}\omega_1\Sigma\,(y^2+z^2)\,\delta m-\omega_2\Sigma\,xy\delta m-\omega_3\Sigma\,zx\delta m&=L\\ \omega_2\Sigma\,(z^2+x^2)\,\delta m-\omega_3\Sigma\,yz\delta m-\omega_1\Sigma\,xy\delta m&=M\\ \omega_3\Sigma\,(x^2+y^2)\,\delta m-\omega_1\Sigma\,zx\delta m-\omega_2\Sigma\,yz\delta m&=N\end{aligned}\right\}\ldots\ldots\ (3).$$

To simplify these equations, suppose the axes of co-ordinates to coincide with the principal axes through the centre of gravity, and let A, B, C, represent the principal moments of inertia of the system: then we have

$$\left.\begin{aligned}A\omega_1&=L\\ B\omega_2&=M\\ C\omega_3&=N\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ (4).$$

Again, for the motion of the centre of gravity we have, if m be the whole mass of the system,

$$\left.\begin{aligned}m\overline{V}_x&=X\\ m\overline{V}_y&=Y\\ m\overline{V}_z&=Z\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ (5);$$

and therefore, for the components of the absolute velocity of any particle δm, we shall have

$$\left.\begin{aligned}V_x&=\overline{V}_x+V_x'=\frac{X}{m}+z\frac{M}{B}-y\frac{N}{C}\\ V_y&=\overline{V}_y+V_y'=\frac{Y}{m}+x\frac{N}{C}-z\frac{L}{A}\\ V_z&=\overline{V}_z+V_z'=\frac{Z}{m}+y\frac{L}{A}-x\frac{M}{B}\end{aligned}\right\}\ldots\ldots\ldots\ (6).$$

Considering V_x, V_y, V_z, as constant, any two of the equations (6) will represent a straight line: multiplying them in order by $\frac{L}{A}$, $\frac{M}{B}$, $\frac{N}{C}$, we get, as a condition to which V_x, V_y, V_z are subject,

$$\frac{L.V_x}{A}+\frac{M.V_y}{B}+\frac{N.V_z}{C}=\frac{L.X}{mA}+\frac{M.Y}{mB}+\frac{N.Z}{mC}\ \ldots\ldots\ldots\ (7).$$

The direction-cosines of the line are, as appears from its equations, proportional to

$$\frac{L}{A},\quad \frac{M}{B},\quad \frac{N}{C}:$$

but, if the line be the spontaneous axis, these cosines, as is evident from the definition, must also be proportional to V_x, V_y, V_z:

hence, putting

$$\left.\begin{aligned} V_x &= \frac{kL}{A} \\ V_y &= \frac{kM}{B} \\ V_z &= \frac{kN}{C} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8);$$

we see, from (7), that

$$k = \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (8) and (9), V denoting the velocity of the spontaneous axis, we see that

$$\begin{aligned} V &= (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}} \\ &= k\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)^{\frac{1}{2}} \\ &= \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)^{\frac{1}{2}}}. \end{aligned}$$

The equations (6) become, by (8),

$$\begin{aligned} \frac{kL}{A} &= \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C}, \\ \frac{kM}{B} &= \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A}, \\ \frac{kN}{C} &= \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B}; \end{aligned}$$

which are the equations to the spontaneous axis, k being supposed to have the value given in the equation (9).

For further information on this problem the student is referred to two papers in the *Cambridge Mathematical Journal*, Vol. IV.

November 1844; the former paper having been contributed by Mr Goodwin.

(8) A given inelastic mass is let fall from a given height on one scale of a balance, and two inelastic masses are let fall from different heights on the other scale, so that the three impacts take place simultaneously; to find the relations between the masses and heights in order that the balance may remain permanently at rest.

If m be the given mass, m', m'', the other two masses, and h, h', h'', respectively the three heights, then

$$m : m' : m'' :: h'^{\frac{1}{2}} - h''^{\frac{1}{2}} : h^{\frac{1}{2}} - h''^{\frac{1}{2}} : h'^{\frac{1}{2}} - h^{\frac{1}{2}}.$$

(9) A uniform horizontal stick, falling by the action of gravity, strikes at one end against a stone; to compare the blow it receives with what it would have received had both ends struck simultaneously against two stones, the blows being supposed to take place at right angles to the stick.

The blow it actually receives is half the blow it would have received, on the latter hypothesis, at each stone.

(10) To determine the nature of the impulses which must be impressed upon a free quiescent cube in order that, *ipso motus initio*, a diagonal of the cube may remain at rest.

Let X, Y, Z, be the components of the resultant force through O, the centre of gravity of the cube, along rectangular axes Ox, Oy, Oz, parallel to the edges; let L, M, N, be the components of the resultant couples about these axes: and let $x = y = z$, be the equations to the quiescent diagonal; then

$$X = 0,\ Y = 0,\ Z = 0,$$

and

$$L = M = N;$$

results which shew that the resultant force must be zero, and that the plane of the resultant couple must be at right angles to the quiescent diagonal.

(11) To determine the nature of the impulses in the preceding problem, in order that an edge of the cube may remain for an instant quiescent.

Let the equations to the quiescent edge be

$$y = a, \qquad z = a,$$

$2a$ being the length of an edge: then

$$X = 0, \qquad M = 0, \qquad N = 0,$$

$$Y = \frac{3L}{2a}, \qquad Z = -\frac{3L}{2a}:$$

results which shew that the resultant force is at right angles to the diagonal plane through the quiescent edge, and that the plane of the resultant couple is perpendicular to this edge.

(12) A lamina, in the form of a semi-ellipse bounded by the axis minor, is moveable about the centre as a fixed point, and falls from the position in which its plane is horizontal:

(1) to determine the impulse which must be applied at the centre of gravity, when the lamina is vertical, in order to reduce it to rest; (2) if this force be applied perpendicularly to the lamina at the extremity of an ordinate through the centre of gravity, instead of being applied at the centre of gravity itself, to ascertain the position of the axis of revolution the instant afterwards.

The required impulse is equal to $M(\frac{8}{3}\pi ga)^{\frac{1}{2}}$, M being the mass of the ellipse, and the required axis is the axis major.

Mackenzie and Walton: *Solutions of the Cambridge Problems for* 1854.

(13) A quiescent ellipsoid is struck by a system of blows, the resultants of which are a force, through its centre of gravity, the direction-cosines of which are l, m, n, and a couple, the direction-cosines of the axes of which are λ, μ, ν: to determine the ratio of the velocity of the spontaneous axis of rotation to the velocity of the centre of gravity of the ellipsoid.

The required ratio is equal to

$$\frac{\dfrac{l\lambda}{b^2+c^2} + \dfrac{m\mu}{c^2+a^2} + \dfrac{n\nu}{a^2+b^2}}{\left\{\dfrac{\lambda^2}{(b^2+c^2)^2} + \dfrac{\mu^2}{(c^2+a^2)^2} + \dfrac{\nu^2}{(a^2+b^2)^2}\right\}^{\frac{1}{2}}}.$$

SECT. 2. *Several Bodies. Smooth Surfaces.*

(1) A heavy sphere P (fig. 244) falls down from a given altitude upon a body at rest, the upper surface of which is a smooth inclined plane; the body is capable of sliding along a smooth horizontal plane, its lower surface being flat; also the same vertical plane contains the centres of gravity both of the sphere and of the body: to determine the initial motions of the sphere and of the body, both of which are supposed to be perfectly inelastic.

Let ABH denote the section of the body made by a plane passing through its centre of gravity, AH being a line in the horizontal plane.

Let V be the velocity of the sphere just before impact; u, v, the resolved parts of its velocity after impact, perpendicular and parallel to the hypotenuse BA of the triangle BAH; u' the velocity of the body parallel to AH after the impact; m, m', the respective masses of the sphere and body, and B the blow of collision; let $\angle BAH = \alpha$.

Then, for the motion of the sphere, the blow being at right angles to BA,

$$mu = mV\cos\alpha - B \quad\text{.................}(1),$$

$$v = V\sin\alpha \quad\text{.........................}(2);$$

and, for the motion of the body,

$$m'u' = B\sin\alpha \quad\text{.......................}(3).$$

These three equations involve four unknown quantities, u, v, u', B: for the solution of the problem, then, another equation will be necessary. This will be obtained by the consideration that, the sphere and the body being both perfectly inelastic, the effect of their collision is merely to prevent the penetration of the one into the interior of the other, without causing any recoil, which could result only from the existence of elasticity: hence the velocity of the ball, after collision, at right angles to the line BA, must be equal to the velocity of any assigned point in this line estimated in the same direction.

Now the velocity of any assigned point in BA at right angles to this line is evidently $u' \sin \alpha$; and therefore we have

$$u' \sin \alpha = u \text{ ....(4)}.$$

From the equations (1), (3), (4), we obtain

$$mB \sin^2 \alpha = mm' V \cos \alpha - m'B,$$

and therefore $$B = \frac{mm' V \cos \alpha}{m \sin^2 \alpha + m'} \text{ ....(5)},$$

which gives the magnitude of the blow.

From (3) and (5), we get

$$u' = \frac{m V \sin \alpha \cos \alpha}{m \sin^2 \alpha + m'},$$

which determines the motion of the body; and therefore, by (4),

$$u = \frac{m V \sin^2 \alpha \cos \alpha}{m \sin^2 \alpha + m'}.$$

D'Arcy; *Mémoires de l'Académie des Sciences de Paris*, 1747, p. 344.

(2) Two billiard balls B and C (fig. 245) are lying in contact on the table: to find the direction in which the ball B must be struck by a third ball A so as to go off in a given direction BD; the balls being of equal volume and weight, and perfectly smooth.

Let the direction AB, which joins the centres of A and B at the instant of their collision, make an angle θ with the straight line CBE, and let $\angle CBD = \alpha$. We will first suppose the balls to be inelastic. Let a, a', denote the resolved parts of the velocity of A in the direction AB, before and after collision respectively; and b, c, the velocities of B, C. Let m represent the mass of each of the balls, R the blow between A, B, and S that between B, C.

Then, for the motion of B, resolving forces at right angles to EC,

$$mb \sin \alpha = R \sin \theta \text{ ....(1)},$$

and, resolving parallel to EC,

$$mb \cos \alpha = R \cos \theta - S \text{ ....(2)}.$$

Also, for the motion of C, we have

$$mc = S \text{..............................} (3).$$

Again, since after collision the velocities of B and C in the direction EC must be equal, there is

$$b \cos \alpha = c \text{..........................} (4).$$

From (3) and (4) we get $mb \cos \alpha = S$,

and therefore, from (2),

$$mb \cos \alpha = R \cos \theta - mb \cos \alpha,$$

$$R \cos \theta = 2mb \cos \alpha \text{..........................} (5).$$

From (1) and (5) there is

$$mb \sin \alpha \cos \theta = 2mb \cos \alpha \sin \theta,$$

and therefore $\qquad \tan \theta = \frac{1}{2} \tan \alpha,$

which determines the point in which A must come into collision with B.

If we introduce the consideration of elasticity, the magnitudes of R and S will each have to be increased in the ratio of $1 + e$ to 1. Now the direction of B's motion will evidently not be affected by any alteration in the absolute magnitudes of R and S, provided that the ratio between their intensities be not changed. Thus we see that the consideration of elasticity will not modify the solution of the problem.

(3) A billiard ball impinges simultaneously upon two other billiard balls which are resting in contact; to determine the motions of the three balls after collision.

Let A (fig. 246) denote the centre of the impinging ball at the moment of impact; A', A'', of the other two balls. Join AA', AA'', and produce them indefinitely to points a', a''; draw Aa a common tangent to the two balls A', A''. Then evidently after collision A's motion will be confined to the straight line Aa; while A', A'', will proceed to move along $A'a'$, $A''a''$. Let u, v, be the velocities of A before and after impact; v' the velocity after impact of each of the balls A', A''. Since $AA'A''$ is evidently an equilateral triangle, $\angle aAa' = \frac{1}{6}\pi$

$=\angle aAa''$; let B be the blow of collision between A, A', and A, A''; and m the mass of each of the three balls. Then

$$mv = mu - 2B\cos\tfrac{1}{6}\pi \quad\text{.................. (1)},$$

$$mv' = B \quad\text{.................................. (2)}.$$

Let us first suppose the three balls to be perfectly inelastic; then the instant after impact the balls A, A', will move in contact, as well as the balls A, A''; hence we must evidently have

$$v' = v\cos\tfrac{1}{6}\pi\,;$$

we have, therefore, from (2),

$$B = mv\cos\frac{\pi}{6}\,;$$

and consequently, from (1),

$$mv = mu - 2mv\cos^2\frac{\pi}{6}, \qquad v = \frac{u}{1+2\cos^2\dfrac{\pi}{6}} = \frac{2u}{5}\,;$$

and therefore $$v' = v\cos\frac{\pi}{6} = \frac{3^{\frac{1}{2}}}{5}u,$$

and $$B = \frac{3^{\frac{1}{2}}mu}{5} \quad\text{.......................... (3)}.$$

If the balls be elastic, and e denote their elasticity, we must increase the value of B in (3) in the ratio of $1+e$ to 1; hence we have

$$B = (1+e)\frac{3^{\frac{1}{2}}mu}{5} \quad\text{..................... (4)},$$

and therefore, from (1),

$$mv = mu - \tfrac{3}{5}(1+e)\,mu,$$

$$v = \tfrac{1}{5}(2-3e)\,u\,;$$

and, from (2), (4), $$mv' = (1+e)\frac{3^{\frac{1}{2}}mu}{5},$$

$$v' = \frac{3^{\frac{1}{2}}}{5}(1+e)\,u.$$

Maclaurin; *Treatise of Fluxions.* D'Alembert; *Traité de Dynamique*, p. 227.

(4) A ball C (fig. 247) impinges upon an inelastic beam AB with a given velocity, at right angles to its length; to determine the magnitude of the blow and the initial circumstances of the motion of the beam and ball.

Let G be the centre of gravity of the beam; m' its mass, k the radius of gyration about G; let $EG = a$, $u =$ the velocity of the ball before impact, $v =$ its velocity immediately afterwards; $R =$ the magnitude of the mutual impulse: let v' be the velocity of G and ω the angular velocity of the beam about G just after collision.

Then, for the initial motion of the ball after collision, m denoting its mass,

$$mv = mu - R \qquad\text{(1)};$$

and, for the initial motion of the beam,

$$m'v' = R \qquad\text{(2)},$$

$$m'k^2\omega = Ra \qquad\text{(3)}.$$

Again, the velocity of the point E of the beam will be equal to

$$v' + a\omega,$$

the former term of the expression being due to the motion of G, and the latter to the rotation about G; but, the beam and ball being inelastic, the velocity of the point E of the beam after collision must be equal to that of the point E of the ball, and therefore of the point C: hence we have

$$v = v' + a \qquad\text{(4)}.$$

From (1), (2), (3), (4), we obtain

$$u - \frac{R}{m} = \frac{R}{m'} + \frac{Ra^2}{m'k^2},$$

and therefore

$$R = \frac{u}{\dfrac{1}{m} + \dfrac{1}{m'} + \dfrac{a^2}{m'k^2}} = \frac{mm'k^2u}{(m + m')\,k^2 + ma^2}.$$

Hence, from (1), we get

$$v = u - \frac{m'k^2u}{(m + m')\,k^2 + ma^2} = \frac{m\,(k^2 + a^2)\,u}{(m + m')\,k^2 + ma^2},$$

from (2), $$v' = \frac{mk^2 u}{(m+m')k^2 + ma^2},$$

and, from (3), $$\omega = \frac{mau}{(m+m')k^2 + ma^2}.$$

(5) A cylinder is revolving with a given angular velocity round its axis, which is horizontal, when it suddenly begins to draw up a weight, consisting of inelastic materials, by means of an inextensible string wound round the cylinder; to determine the time the system will continue in motion, and the original distance of the weight from the cylinder, that, at the instant the motion ceases, the weight may just touch the cylinder.

Let a = the radius of the cylinder, m = its mass, k^2 = the radius of gyration about its axis; m' = the mass of the weight; let ω, ω', denote the angular velocities of the cylinder just before and just after beginning to draw up the weight; let u = the velocity of the weight at the commencement of its motion, B = the impulsive force exerted initially by the string on the weight.

Then we shall have

$$\omega' = \omega - \frac{Ba}{mk^2}, \qquad u = \frac{B}{m'};$$

but $u = a\omega'$; hence

$$\frac{B}{m'} = a\omega - \frac{Ba^2}{mk^2},$$

and therefore

$$B = \frac{mm'a\omega k^2}{m'a^2 + mk^2}, \qquad u = \frac{ma\omega k^2}{m'a^2 + mk^2} \ldots\ldots\ldots\ldots\ldots (1).$$

Let θ denote the angle through which the cylinder has revolved about its axis at the end of the time t from the commencement of the raising of the weight; and let x be the corresponding distance of the weight below the horizontal plane through the axis of the cylinder. Then, by the Principle of the Conservation of Vis Viva,

$$m'\frac{dx^2}{dt^2} + mk^2\frac{d\theta^2}{dt^2} = 2m'gx + C \ldots\ldots\ldots\ldots (2):$$

but, if b denote the value of x at the commencement of the raising of the weight, it is clear from the geometry that

$$x + a\theta = b, \quad \text{and therefore } a\frac{d\theta}{dt} = -\frac{dx}{dt};$$

hence, from (2),

$$\frac{m'a^2 + mk^2}{a^2}\frac{dx^2}{dt^2} = 2m'gx + C;$$

differentiating with respect to t, and dividing by $2\dfrac{dx}{dt}$,

$$(m'a^2 + mk^2)\frac{d^2x}{dt^2} = m'a^2g;$$

integrating with respect to t, we have

$$(m'a^2 + mk^2)\frac{dx}{dt} = C + m'a^2gt;$$

but $\dfrac{dx}{dt} = -u$ when $t = 0$; hence $C = -(m'a^2 + mk^2)u$, or, by (1), $C = -ma\omega k^2$, and therefore

$$(m'a^2 + mk^2)\frac{dx}{dt} = m'a^2gt - ma\omega k^2 \;\ldots\ldots\ldots\ldots\; (3);$$

integrating again with respect to t, we get

$$(m'a^2 + mk^2)x = C' + \tfrac{1}{2}m'a^2gt^2 - ma\omega k^2t;$$

but $x = b$ when $t = 0$; hence $C' = (m'a^2 + mk^2)b$, and therefore

$$(m'a^2 + mk^2)x = (m'a^2 + mk^2)b + \tfrac{1}{2}m'a^2gt^2 - ma\omega k^2t \;\ldots\ldots\; (4).$$

Let t' denote the time when the motion ceases for an instant; then, from (3), since $\dfrac{dx}{dt} = 0$ when $t = t'$,

$$0 = m'a^2gt' - ma\omega k^2, \quad t' = \frac{m\omega k^2}{m'ag}.$$

Hence also, from (4), since $x = 0$ when $t = t'$,

$$(m'a^2 + mk^2)b = ma\omega k^2t' - \tfrac{1}{2}m'a^2gt'^2 = \frac{m^2\omega^2k^4}{2m'g},$$

which gives the required value of b.

(6) Two inelastic spheres, of which A and a (fig. 248) are the centres, are attached to rigid rods CA and ca, which are

capable of motion in a single plane about axes through C and c at right angles to the plane; supposing the spheres to impinge against each other with given velocities, it is required to determine their initial velocities after impact.

Join Aa, and produce it indefinitely both ways to points α, β; from C, c, draw CG, cg, at right angles to $\alpha\beta$ at the moment of collision. Let C, c, represent the lines CG, cg, Ω, ω, the angular velocities of CA, ca, about C, c, respectively immediately before, and Ω', ω', immediately after collision, the angular motions being estimated in the directions indicated by the arrows in the figure; B the blow of collision; I the moment of inertia of the sphere A with its rod AC about the axis through C, i the moment of inertia of the other sphere and rod about c.

Then, $\Omega'-\Omega$ being the angular velocity which CA gains, and $\omega-\omega'$ that which ca loses by the shock, we shall have

$$I(\Omega'-\Omega)=B.\,C, \qquad i(\omega-\omega')=B.\,c \ldots\ldots\ldots\ldots (1).$$

But, the spheres being inelastic, the point P of the sphere a will, the instant after collision, have the same velocity along $\alpha\beta$ as the point P of the sphere A: hence

$$\Omega'.\,CP.\sin\angle CPG=\omega'.\,cP.\sin\angle cPg,$$

or

$$\Omega'C=\omega'c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now, from the equations (1),

$$cI(\Omega'-\Omega)+Ci(\omega'-\omega)=0,$$

and therefore, by (2), we get

$$cI(c\omega'-C\Omega)+C^2i(\omega'-\omega)=0,$$

$$(c^2I+C^2i)\,\omega'=C(cI\Omega+Ci\omega),$$

and therefore $\quad (c^2I+C^2i)\,\Omega'=c(cI\Omega+Ci\omega)$;

which two equations give the values of Ω', ω'.

The solution of a particular case of this problem was unsuccessfully attempted by John Bernoulli, son of the celebrated John Bernoulli, in the *Mémoires de St. Pétersbourg*, Tom. VII.: the correct solution of the problem, in its most general form,

was given by D'Alembert, *Traité de Dynamique*, p. 221; second edition.

(7) An inelastic sphere (A) slides down an inclined plane and comes into contact with an equal inelastic sphere (B) lying on a horizontal plane and also in contact with the inclined plane; to determine the velocities of the two spheres just after collision, and the angular velocity of the line joining the centres of A and B the instant before its becoming horizontal.

Let l = the length of the portion of the inclined plane down which the sphere A has slid from rest before collision with B, α = the inclination of the plane, r = the radius of either sphere; let v, v', be the velocities of A, B, respectively, just after collision, and ω the required angular velocity of the distance between their centres: then

$$v = \frac{\cos^2\alpha}{1+\cos^2\alpha}\cdot(2gl\sin\alpha)^{\frac{1}{2}}, \qquad v' = \frac{\cos\alpha}{1+\cos^2\alpha}\cdot(2gl\sin\alpha)^{\frac{1}{2}},$$

$$\omega^2 = \frac{g\sin^3\alpha}{2r^2}\cdot\frac{(l+2r)\cos^2\alpha+2r}{(1+\cos^2\alpha)^2}.$$

(8) A uniform bar, moveable in a vertical plane about one of its ends, falls from a horizontal position and strikes a perfectly elastic ball: to determine the greatest velocity which it can communicate to the ball, and to find the position in which the ball must be struck to receive this velocity.

Let a denote the length of the bar, m, m', the masses of the bar and ball respectively; then the greatest velocity which can be communicated to the ball is equal to $\left(ga\cdot\frac{m}{m'}\right)^{\frac{1}{2}}$; and, in order to acquire this velocity, the ball must be placed at a distance, vertically below the point of suspension, equal to $a\sqrt{\left(\frac{m}{3m'}\right)}$.

(9) Two equal inelastic balls, connected by a rigid rod without weight, slide down in a vertical plane between a smooth vertical and a smooth horizontal plane: if the lower ball impinge directly upon an equal ball at rest, to determine the angular velocity of the rod just after the collision, its angular velocity and position just before collision being known.

If α denote the inclination, and ω the angular velocity of the rod just before the collision, its angular velocity just afterwards will be equal to

$$\frac{\omega}{1+\sin^2\alpha}.$$

(10) A rectangular door, which is open, is struck perpendicularly at the outer edge by a body, the mass of which is one third of that of the door, and which is moving with a given velocity: to determine the velocity communicated to the outer edge of the door, the hinges being supposed smooth and the colliding bodies inelastic.

The outer edge of the door will acquire a velocity equal to half that of the impinging body.

(11) A slender homogeneous rod lies on a smooth horizontal plane: it is divided into two portions by a joint at its middle point: it is set in motion by a blow at one extremity perpendicular to its length: to compare the initial velocity of the middle point of the rod with that which it would have had supposing the rod had not been jointed.

The velocity of the middle point of the rod, when jointed, is twice as great as if it had been left in one piece, and in the opposite direction.

(12) A beam, placed upon a smooth horizontal plane, has one extremity fixed: a ball A is placed on the plane in contact with the beam at a given distance from the fixed extremity: to determine at what point of the beam, on the other side of it, another ball A' must impinge directly, so that the greatest possible velocity may be communicated to A by the impact, the beam and balls being inelastic.

Let a, a', be the distances, at the instant of impact, of A, A', respectively, from the fixed end of the beam; m, m', the respective masses of A, A'; and μk^2 the moment of inertia of the beam about its fixed end. Then

$$a'=\left\{\frac{1}{m'}(ma^2+\mu k^2)\right\}^{\frac{1}{2}}.$$

O'Brien and Ellis: *Solutions of the Senate-House Problems for* 1844.

(13) A perfectly inelastic and smooth ellipsoid the semi-axes of which are a, b, c, revolving with an angular velocity ω round one axis c, impinges with a velocity v upon a quiescent sphere of equal magnitude: the instant before collision, the semi-axis a lies in the direction of the motion of the centre of gravity of the ellipsoid: at the instant of impact, the sphere touches the ellipsoid at the extremity of the latus rectum of its principal section containing a and b: supposing the eccentricity of that principal section to be equal to $\sqrt{\frac{2}{3}}$, to determine the relation between v and ω that there may be no rotatory motion in the ellipsoid after collision.

The required relation is

$$\frac{v}{\omega} = 2a.$$

(14) A rectangular door of weight W is closed by means of a weight W', suspended at one end of a chord, which passes over a pulley at the edge of the door when shut: the cord winds on and off the arc of a circle the radius of which is equal to the breadth of the door: to determine the angular velocity of the door the instant before and the instant after arriving at its position of equilibrium.

Let $\alpha =$ the angular distance of the door, in a position of instantaneous rest, from its position of equilibrium, $b =$ the breadth of the door; and let ω, ω', be respectively the required angular velocities: then

$$\omega^2 = \frac{6W'g\alpha}{b(W + 3W')}, \qquad \omega' = \frac{W - 3W'}{W + 3W'} \cdot \omega.$$

If $W = 3W'$, it appears that the door will finally rest in its position of equilibrium.

SECT. 3. *Rough Surfaces.*

(1) An inelastic cylinder O (fig. 249) having rolled down a perfectly rough plane CA, impinges upon a perfectly rough plane $C'A$, the axis of the cylinder being parallel to the

intersection of the two planes; to find the velocity with which the cylinder will commence its ascent up the second plane, and the limiting angle of inclination of the two planes for which the ascent is possible.

Let $\angle CAC' = \alpha$, $k =$ the radius of gyration of the cylinder about its axis, $a =$ the radius of the cylinder, $m =$ its mass; $u =$ the velocity of the centre O of a circular section of the cylinder parallel to the plane CA just before impact, and $v =$ the velocity after impact up the plane AC'; $R =$ the impulsive force of friction exerted by the plane AC' upon the cylinder at the moment of impact to secure perfect rolling.

Then, for the motion of the centre of gravity of the cylinder parallel to AC', we have

$$mv = R - mu \cos \alpha \;\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, for the value of the decrement of the angular velocity of the cylinder about its axis owing to the impulse R, we have the expression

$$\frac{Ra}{mk^2},$$

which, by (1), is equal to

$$\frac{a\,(v + u \cos \alpha)}{k^2};$$

but, the planes being both perfectly rough, it is evident that $\frac{u}{a}$ is the angular velocity before, and $\frac{v}{a}$ after the impact; hence

$$\frac{u}{a} - \frac{v}{a} = \frac{a\,(v + u \cos \alpha)}{k^2};$$

$$u - v = \frac{a^2}{k^2}(v + u \cos \alpha);$$

but $a^2 = 2k^2$; hence

$$u - v = 2v + 2u \cos \alpha;$$

and therefore, $\quad v = \frac{1}{3}(1 - 2 \cos \alpha)\, u,$

which gives the velocity with which the cylinder ascends the plane AC'.

Since v can evidently, from the nature of the case, have neither a zero nor a negative value, it is clear that the smallest possible value of α for the ascent is given by the equation

$$1 - 2\cos\alpha = 0, \quad \cos\alpha = \tfrac{1}{2};$$

whence $\frac{1}{3}\pi$ is the limiting value of α.

(2) A ball, sliding without rotation along a smooth horizontal plane, impinges obliquely against a perfectly rough vertical plane; to determine the subsequent motion of the ball.

Let Ox, Oy, Oz, (fig. 250), be three rectangular axes, the plane xOy being horizontal and passing through the centre C of the ball, and the plane xOz being the rough vertical plane against which the ball impinges. Let E be the point in which the ball strikes against the vertical plane; CF the direction of the motion of C before collision. Let u be the velocity of C before impact, α the inclination of CF to Ox; v_x, v_y, the resolved parts of the velocity of C parallel to Ox, Oy, after the impact; ω the angular velocity of the ball about C after impact; X, Y, the impulsive reactions of the rough plane along xO, and parallel to Oy, during collision; m the mass of the ball; a the radius of the ball, k the radius of gyration about C.

First we will suppose the ball to be inelastic. For the motion of the ball after impact, we have

$$mv_x = mu\cos\alpha - X \quad\ldots\ldots\ldots\ldots(1);$$

$$mv_y = Y - mu\sin\alpha \quad\ldots\ldots\ldots\ldots(2),$$

$$mk^2\omega = Xa \quad\ldots\ldots\ldots\ldots(3).$$

Now, the ball being perfectly inelastic, the velocity of C at right angles to the vertical plane will be destroyed by the impact, or $v_y = 0$; hence, from (2),

$$Y = mu\sin\alpha \quad\ldots\ldots\ldots\ldots(4).$$

Also, the vertical plane being perfectly rough, the ball must roll without sliding after impact; hence we must have $a\omega = v_x$, and therefore, from (1), (3), we get

$$k^2(mu\cos\alpha - X) = Xa^2,$$

$$X = \frac{mk^2u\cos\alpha}{a^2 + k^2} \quad\ldots\ldots\ldots\ldots(5).$$

Next, let us suppose the ball to be elastic, e denoting the elasticity; then, v_x', v_y', ω', denoting on the new supposition what v_x, v_y, ω, were taken to denote on the old one, we shall have

$$mv_x' = mu\cos\alpha - (1+e)\,X \;\ldots\ldots\ldots\ldots\; (6),$$

$$mv_y' = mu\sin\alpha - (1+e)\,Y \;\ldots\ldots\ldots\ldots\; (7),$$

$$mk^2\omega' = (1+e)\,Xa \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8).$$

From the equations (5) and (6), we obtain

$$v_x' = u\cos\alpha - \frac{(1+e)\,k^2u\cos\alpha}{a^2+k^2} = \frac{a^2-ek^2}{a^2+k^2}\,u\cos\alpha\,;$$

from (4) and (7),

$$v_y' = u\sin\alpha - (1+e)\,u\sin\alpha = -eu\sin\alpha\,;$$

and, from (5), (8),

$$\omega' = \frac{(1+e)\,au\cos\alpha}{a^2+k^2}\,;$$

which values of v_x', v_y', ω', completely determine the subsequent motion of the ball.

(3) A perfectly rough sphere is placed upon a perfectly rough horizontal plane which is made to rotate with a uniform angular velocity about a vertical axis; to determine the path described by the sphere in space.

Let Oz (fig. 251) be the vertical axis about which the plane revolves; let Ox, Oy, be any two horizontal lines fixed in space and at right angles to each other; P the point of contact of the sphere with the revolving plane at any time t; draw PM parallel to yO. Let $OM = x$, $PM = y$; $a =$ the radius of the sphere, $m =$ its mass, $mk^2 =$ its moment of inertia about a diameter; $\omega =$ the angular velocity of the revolving plane about Oz, the motion being supposed to take place in the direction indicated by the arrow in the plane xOy in the figure; let X, Y, denote the resolved parts of the friction exerted by the plane on the sphere, estimated parallel to Ox, Oy, respectively; ω', ω'', the angular velocities of the sphere about diameters parallel to Ox, Oy, the directions of these velocities being estimated in the manner indicated by the arrows in the planes yOz, zOx.

For the motion of the centre of gravity of the sphere we have

$$m\frac{d^2x}{dt^2} = X \text{..............................} (1),$$

$$m\frac{d^2y}{dt^2} = Y \text{..............................} (2);$$

and, for the rotation of the sphere,

$$mk^2\frac{d\omega'}{dt} = Ya \text{..........................} (3),$$

$$mk^2\frac{d\omega''}{dt} = -Xa \text{........................} (4).$$

From (1) and (4) we have, eliminating X,

$$a\frac{d^2x}{dt^2} = -k^2\frac{d\omega''}{dt} \text{......................} (5);$$

and, from (2), (3), eliminating Y,

$$a\frac{d^2y}{dt^2} = k^2\frac{d\omega'}{dt} \text{........................} (6).$$

Integrating the equations (5), (6), and adding arbitrary constants, we have

$$a\frac{dx}{dt} = C - k^2\omega'' \text{.....................} (7),$$

$$a\frac{dy}{dt} = C' + k^2\omega' \text{....................} (8).$$

Now the linear velocity of the centre of gravity of the sphere relatively to the rough plane, in consequence of the rolling of the sphere, will be

$a\omega''$ parallel to Ox, $-a\omega'$ parallel to Oy;

and the linear velocity due to the rotation of the rough plane will be

$-\omega y$ parallel to Ox, ωx parallel to Oy;

hence, $\frac{dx}{dt}$, $\frac{dy}{dt}$, being the whole linear velocity of the centre of the sphere parallel to Ox, Oy, respectively, we have

$$\frac{dx}{dt} = -\omega y + a\omega'', \quad \frac{dy}{dt} = \omega x - a\omega';$$

and therefore, by the aid of (7) and (8), eliminating ω'' and ω',

$$a\frac{dx}{dt} = C - \frac{k^2}{a}\left(\omega y + \frac{dx}{dt}\right), \qquad \left(1+\frac{a^2}{k^2}\right)\frac{dx}{dt} = \frac{aC}{k^2} - \omega y,$$

$$a\frac{dy}{dt} = C' + \frac{k^2}{a}\left(\omega x - \frac{dy}{dt}\right), \qquad \left(1+\frac{a^2}{k^2}\right)\frac{dy}{dt} = \frac{aC'}{k^2} + \omega x;$$

eliminating t between these two equations, we get

$$(aC - k^2\omega y)\,dy = (aC' + k^2\omega x)\,dx:$$

integrating and adding an arbitrary constant C'', we obtain

$$2aCy - k^2\omega y^2 = 2aC'x + k^2\omega x^2 + C'',$$

or
$$x^2 + y^2 + \frac{2aC'}{k^2\omega}x - \frac{2aC}{k^2\omega}y + \frac{C''}{k^2\omega} = 0 \text{............ (9).}$$

We proceed now to the determination of the arbitrary constants. Let the initial distance of the centre of the sphere from the axis of z be b, and let the axis of x be so chosen as to pass through the initial position of the point of contact of the sphere with the rough plane. Then, since the initial impulse of the friction of the revolving plane upon the sphere is at right angles to the axis of x, we shall have initially $\frac{dx}{dt} = 0$, $\omega'' = 0$; hence, from (7), we see that $C = 0$. Again, F denoting the impulse of the friction when the sphere is just placed upon the revolving plane, and $\left(\frac{dy}{dt}\right)$, (ω'), denoting the values of $\frac{dy}{dt}$, ω', just after the impulse, we shall have

$$m\left(\frac{dy}{dt}\right) = F \text{.......................... (10),}$$

$$mk^2(\omega') = Fa \text{......................... (11).}$$

But, since there is no sliding between the sphere and the plane, it is clear that

$$\left(\frac{dy}{dt}\right) = b\omega - a(\omega') \text{.................... (12),}$$

where $b\omega$ is the velocity of the centre of the sphere parallel to Ox due to the rotation of the plane, and $-a(\omega')$ the velocity

estimated in the same direction due to the rolling of the sphere along the plane: hence from (10) we have

$$m\{b\omega - a(\omega')\} = F,$$

and therefore, by (11),

$$a\{b\omega - a(\omega')\} = k^2(\omega'), \quad (\omega') = \frac{ab\omega}{a^2 + k^2};$$

and then, by (12),

$$\left(\frac{dy}{dt}\right) = b\omega - \frac{a^2b\omega}{a^2 + k^2} = \frac{k^2b\omega}{a^2 + k^2}.$$

But from (8) we have

$$a\left(\frac{dy}{dt}\right) = C' + k^2(\omega');$$

hence, putting for $\left(\frac{dy}{dt}\right)$ and (ω') their values,

$$\frac{abk^2\omega}{a^2 + k^2} = C' + \frac{abk^2\omega}{a^2 + k^2}, \quad C' = 0.$$

Since $C = 0$ and $C' = 0$, we have, from (9),

$$x^2 + y^2 + \frac{C''}{k^2\omega} = 0;$$

but $x = b$ when $y = 0$, and therefore

$$b^2 + \frac{C''}{k^2\omega} = 0,$$

and we get for the equation to the path of the centre of the sphere in space

$$x^2 + y^2 = b^2,$$

the equation to a circle having O for its centre.

The following elegant solution of this problem has been communicated to me by Mr R. Leslie Ellis:

"A sphere, resting on a perfectly rough horizontal plane, receives a tangential impulse when the plane is made to move in its own plane. This impulse gives a velocity to the centre of the sphere and produces an angular velocity about a horizontal axis. The

centre of the sphere moves parallel to the impulse, the axis of rotation is perpendicular to it; therefore the point of contact moves parallel to the impulse and therefore to the direction of motion of the centre. Therefore, as there is no sliding, the centre moves in the same direction as that of the motion of the plane supposed rectilineal. Moreover it is easily seen that the velocity of the centre is to that of the point of contact, or, which is the same thing, to that of the plane, as $1 : 1 + \frac{a^2}{k^2}$, a being the radius of the sphere, k its least radius of gyration. While the direction and velocity of the plane's motion remain unaltered, no farther action occurs; when a change takes place, a new tangential impulse is given to the sphere, producing a new velocity of the centre parallel to its own direction, and a new velocity of rotation about an axis at right angles to it. The new velocity of rotation bearing to the old the same ratio as the new velocity of the centre to the old, the result is a compound velocity of the centre bearing the same ratio as before to the velocity of the point of contact, and as before parallel to it, and therefore still parallel to the direction of motion of the plane; and so on; whether the motion of the plane varies continuously or discontinuously, in direction, in velocity, or in both. In the case proposed the motion of the plane (by which throughout I mean the element thereof in contact with the sphere) is always normal to a line drawn to a fixed point. Therefore the motion of the centre is so too, therefore the centre describes a circle whose centre is perpendicularly over the said fixed point. Q.E.D."

(4) An inelastic homogeneous cylinder rolls down a perfectly rough inclined plane which terminates in a perfectly rough horizontal plane; to find the velocity of the cylinder along the horizontal plane, and the blow which it receives when it first impinges upon it.

Let $\alpha =$ the inclination of the inclined plane to the horizon, $m =$ the mass of the cylinder, $u =$ the velocity of its axis the instant before and u' the instant after impact, $F =$ the initial impulse

of the friction of the horizontal plane, estimated in the direction of the sphere's motion along it, and $B =$ the normal impulse of the horizontal plane; then

$$u' = \tfrac{1}{3}(1 + 2\cos\alpha)\,u, \quad F = \tfrac{1}{3}mu\,(1 - \cos\alpha), \quad B = mu\sin\alpha.$$

(5) A homogeneous cylinder slides, without rolling, down an inclined plane which is for a certain space quite smooth, and, after acquiring a given velocity, is suddenly caused by the roughness of the surface to roll without sliding: to determine the velocity of the axis of the cylinder the instant rolling commences, and to find the initial impulse of friction.

If u be the velocity of the cylinder the instant before and u' the instant after the commencement of perfect rolling, m the mass of the cylinder, F the initial impulse of friction; then

$$u' = \tfrac{2}{3}u, \quad F = \tfrac{1}{3}mu.$$

(6) Two wheels, revolving uniformly in the same plane, are suddenly brought into contact, and their axes are kept fixed; to determine what alteration will take place in their angular velocities, the friction being sufficient to prevent all sliding.

Let $m =$ the mass of one wheel, $k =$ its radius of gyration about its axis of rotation, $a =$ its radius; let α be its angular velocity before and α' after collision. Let n, l, b, β, β', denote like quantities in relation to the other wheel. Then, the revolutions of the two wheels being supposed opposite in character before being brought into contact,

$$\alpha' - \alpha = nal^2 \cdot \frac{b\beta - a\alpha}{mb^2k^2 + na^2l^2},$$

and

$$\beta' - \beta = mbk^2 \cdot \frac{a\alpha - b\beta}{na^2l^2 + mb^2k^2}.$$

(7) A book $ABCD$ is placed, in a vertical plane, with one angle A on a table; to find the greatest ratio which the side BC can bear to the side AB that, after the impact, the book may not tilt over the angle B, the table being supposed to be perfectly rough and the book to be inelastic.

The ratio of BC to AB cannot possibly be greater than $\frac{1}{\sqrt{2}}$: how much less the ratio should be is indeterminate, being dependent upon the physical nature of the contact between the surfaces of the book and table.

(8) A spherical ball of given elasticity, moving with a given velocity, and revolving uniformly round a horizontal axis through its centre and perpendicular to the plane of the motion of its centre, impinges upon a horizontal plane of such a nature as to prevent all sliding: to determine whether the angle of reflection from the plane is increased or diminished by increasing the velocity of rotation before impact; and to find how many revolutions the ball will make before it strikes the plane after impact.

Let the angles of incidence and reflection be α, α', respectively, and conceive the rotation to be estimated in the direction indicated by the arrows in the diagram: fig. (252): let $r =$ the radius of the ball: let u, v, be the components of the velocity of incidence, parallel and perpendicular to the plane, and ω the angular velocity, the instant before impact.

If ω, being positive, be increased, α' will increase. If ω be negative, or the rotation of an opposite character to that indicated in the figure, then α' will decrease as ω increases: if

$$\omega = -\frac{5v}{2r}, \quad \alpha' = 0,$$

or the ball will rebound in the normal: if ω be a greater negative quantity than $-\frac{5v}{2r}$, α' will be negative or the angle of reflection will be on the same side of the normal as the angle of incidence.

The required number of revolutions will be equal to

$$\frac{4\pi eu}{g} \cdot \frac{7r}{2\omega r + 5v}.$$

(9) A perfectly rough plane, moving with a certain velocity parallel to four of the edges of a rigid inelastic cube placed upon

it, is suddenly brought to rest; to determine the velocity in order that the cube may just turn over its edge.

If $c =$ the length of each edge, and $v =$ the required velocity,

$$v^2 = \tfrac{8}{3} (\sqrt{2} - 1)\, cg.$$

(10) An imperfectly elastic homogeneous rough sphere is projected obliquely, without rotation, against a fixed plane: to determine ρ, the ratio of the tangential forces of restitution and compression, in terms of α, α', the angles of incidence and reflection, and e, the coefficient of elasticity for direct impact.

The value of ρ is given by the equation

$$2\rho = 5 - 7\, e \tan \alpha' \cot \alpha.$$

Ferrers and Jackson: *Solutions of the Cambridge Problems*, 1848 to 1851.

(11) A series of perfectly rough semicylinders are fixed, side by side, upon their flat faces directly across a straight road of constant inclination: to determine the inclination of the road in order that a rough circular inelastic hoop, just started downwards from the summit of one of the cylindrical ridges, may travel directly along the road with a uniform mean velocity.

Let $a =$ the radius of the hoop, $a_1 =$ that of one of the cylinders, $\beta =$ the inclination of the road: then, α being given by the formula

$$\sin \alpha = \frac{a_1}{a + a_1},$$

β is determined by the equation

$$\cos^2 \alpha = \frac{\sin \dfrac{\alpha - \beta}{2}}{\sin \dfrac{\alpha + \beta}{2}}.$$

Mackenzie and Walton: *Solutions of the Cambridge Problems for* 1854.

CHAPTER XIII.

LIVE THINGS.

(1) A FLEA is resting on a needle AB at a given point E: the needle lies on a smooth table: the flea then skips to a given point F of the needle; to determine the least initial velocity of the flea.

Let V be the velocity with which the flea skips, α the inclination of V to the horizon, u the velocity of the needle during the flight of the flea, t the time of flight, m, m', the masses of the needle and flea respectively, and let $EF = c$.

Then, since the centre of gravity of the flea and needle will not be affected by the skip,

$$-mu + m'V\cos\alpha = 0 \quad\text{.....................(1).}$$

Also $$V\sin\alpha = \tfrac{1}{2}gt \quad\text{.................................(2).}$$

Now the whole range of the flea is equal to the distance EF diminished by the space through which F has slid backwards during the time of flight; and therefore

$$\frac{V^2\sin 2\alpha}{g} = c - ut = c - \frac{m'}{m}V\cos\alpha \,.\, t, \text{ by (1),}$$

$$= c - \frac{m'V^2\sin 2\alpha}{mg}, \text{ by (2):}$$

hence $$V^2 = \frac{mcg}{m+m'} \,.\, \operatorname{cosec} 2\alpha.$$

The least possible value of V is therefore equal to

$$\left(\frac{mcg}{m+m'}\right)^{\frac{1}{2}}.$$

(2) A beetle, placed upon a movable inclined plane, which rests upon a smooth horizontal plane, sets off to crawl up it at a given uniform velocity relatively to the plane; to determine the velocity of the plane and the pressure exerted upon it by the beetle.

Let P (fig. 253) be the position of the beetle at any time on the inclined plane AB; Ox the smooth horizontal plane; let $\angle BAx = \alpha$, $u =$ the uniform relative velocity of the beetle; $OA = x$, $V = \frac{dx}{dt}$; let N, T, be the impulsive and N', T', the finite actions between the plane and beetle, m, m', being their respective masses.

Since the centre of gravity of the beetle and plane can have no horizontal motion, we have

$$mV + m'(V + u\cos\alpha) = 0,$$

and therefore

$$V = -\frac{m'u\cos\alpha}{m+m'},$$

which shews that the plane travels in the direction xO with a velocity equal to

$$\frac{m'u\cos\alpha}{m+m'}.$$

Again, for the impulsive actions, resolving parallel and perpendicularly to the plane,

$$T = m'(u + V\cos\alpha)$$
$$= m'u \cdot \frac{m + m'\sin^2\alpha}{m+m'},$$

and

$$N = -m'V\sin\alpha$$
$$= \frac{m'^2 u\sin\alpha\cos\alpha}{m+m'},$$

For the finite actions, since the beetle has no acceleration,

$$T' = m'g\sin\alpha, \qquad N' = m'g\cos\alpha.$$

(3) A monkey is put at the top of a pole, which is then placed with its lower end on a smooth horizontal plane, its higher resting against a smooth vertical wall: the monkey contrives to clamber down the pole in such a manner that the pole remains quiescent: to determine the velocity of the monkey when he gets to the lower end of the pole.

Let P (fig. 254) be the position of the monkey on the pole AB at any time t after the commencement of the motion: let m, m', be the masses of the pole and monkey respectively; let R, S, be the pressures exerted by the two planes on the ends of the pole; let N, T, be the normal and longitudinal actions between the pole and the monkey; let $\alpha=$ the inclination of AB to the vertical, $AB=2a$, $AP=x$.

Then, for the motion of the monkey along the pole,

$$m'\frac{d^2x}{dt^2}=T+m'g\cos\alpha \quad\text{.................}\quad (1),$$

and, for the preservation of its contact with the pole,

$$N=m'g\sin\alpha \quad\text{............................}\quad (2).$$

For the equilibrium of the pole, resolving vertically,

$$S+T\cos\alpha=N\sin\alpha+mg \quad\text{.................}\quad (3),$$

and, taking moments about A,

$$Nx+mga\sin\alpha=2aS\sin\alpha \quad\text{...................}\quad (4).$$

From (2), (3), (4), we may readily ascertain that

$$T=m'g\frac{\sin^2\alpha}{\cos\alpha}-\frac{m'gx}{2a\cos\alpha}+\frac{mg}{2\cos\alpha}.$$

Substituting this value of T in (1), we shall get

$$\frac{d^2x}{dt^2}+\frac{gx}{2a\cos\alpha}=\frac{g(m+2m')}{2m'\cos\alpha} \quad\text{.................}\quad (5):$$

multiplying by $2\frac{dx}{dt}$ and integrating, we have, observing that $\frac{dx}{dt}=0$ when $x=0$,

$$\frac{dx^2}{dt^2} = \frac{gx}{2\cos\alpha}\left\{\frac{2}{m'}(m+2m') - \frac{x}{a}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

When therefore $x = 2a$, or the monkey arrives at B, its velocity is equal to

$$\left\{\frac{2ga\,(m+m')}{m'\cos\alpha}\right\}^{\frac{1}{2}}.$$

(4) A needle is suspended from its higher extremity; at the point of suspension there is a spider, the mass of which is equal to that of the needle: supposing the needle to be placed horizontally, and then to be projected downwards with a given angular velocity, to determine how long the spider will take to run to the lower end of the needle, the motion of the spider being such that the angular velocity of the needle may remain unchanged.

Let m = the mass of the needle or spider; let P = the place of the spider at any time on the needle CA, (fig. 255) C being the point of suspension; let N, T, be the normal and longitudinal actions between the needle and spider: draw Cx vertically, and let $CP = x$, $CA = 2a$, $\angle ACx = \theta$; let ω = the angular velocity of the needle.

Then, since the needle's angular velocity is to be constant, we have

$$mga\sin\theta + Nx = 0.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, for the motion of the spider,

$$m\frac{d^2}{dt^2}(x\cos\theta) = mg - N\sin\theta - T\cos\theta\ldots\ldots(2),$$

and

$$m\frac{d^2}{dt^2}(x\sin\theta) = N\cos\theta - T\sin\theta\ldots\ldots\ldots\ldots(3).$$

From (2) and (3) we see that

$$m\left\{x\sin\theta\frac{d^2}{dt^2}(x\cos\theta) - x\cos\theta\frac{d^2}{dt^2}(x\sin\theta)\right\} = mgx\sin\theta - Nx:$$

but

$$x\sin\theta\frac{d^2}{dt^2}(x\cos\theta) - x\cos\theta\frac{d^2}{dt^2}(x\sin\theta)$$

$$= \frac{d}{dt}\left\{x \sin\theta \frac{d}{dt}(x\cos\theta) - x\cos\theta \frac{d}{dt}(x\sin\theta)\right\}$$

$$= -\frac{d}{dt}\left(x^2 \frac{d\theta}{dt}\right):$$

hence $$m\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right) = Nx - mgx \sin\theta,$$

and therefore, by (1), observing that $\frac{d^2\theta}{dt^2} = 0$ by the hypothesis,

$$2x\frac{dx}{dt}\frac{d\theta}{dt} = -g(x+a)\sin\theta:$$

but $\theta = \frac{1}{2}\pi - \omega t$, by the hypothesis: hence

$$2\omega x\frac{dx}{dt} = g(x+a)\cos\omega t,$$

$$\cos\omega t \,.\, dt = \frac{2\omega}{g} \,.\, \frac{xdx}{x+a};$$

integrating and bearing in mind that $x = 0$ when $t = 0$,

$$\frac{1}{\omega}\sin\omega t = \frac{2\omega}{g}\left\{x - a\log\frac{x+a}{a}\right\},$$

and therefore, for the required time,

$$\sin\omega t = \frac{2\omega^2}{g}(2a - a\log 3),$$

$$t = \frac{1}{\omega}\sin^{-1}\left\{\frac{2\omega^2 a}{g}(2 - \log 3)\right\}.$$

This problem may be solved also more briefly thus:

By D'Alembert's Principle we have, taking moments about the point of suspension for the system composed of the spider and needle,

$$\frac{d}{dt}\left(mx^2\frac{d\theta}{dt} + \frac{4}{3}ma^2\frac{d\theta}{dt}\right) = -mg\sin\theta(x+a),$$

and therefore, replacing $\frac{d\theta}{dt}$ by its constant value $-\omega$,

$$2\omega x\frac{dx}{dt} = g(x+a)\cos\omega t,$$

the differential equation obtained by the former method.

(5) A fly is sitting upon a needle, which rests upon a smooth horizontal plane; supposing the fly to walk along the needle to its other end, how far would the needle be displaced?

If c be the length of the needle, and m, m', the masses of the needle and fly respectively, the displacement of the needle will be equal to

$$\frac{m'c}{m+m'}.$$

(6) Two beetles are standing upon the ends of a horizontal rod, which is supported on a smooth table; supposing one beetle to crawl to the middle of the rod, how far must the other crawl along it in order that the final position of the rod may be the same as its original one?

Let β be the mass of the former and β' of the latter beetle, and a the length of the rod. Then, if x be the distance the latter beetle should crawl,

$$x = \frac{a\beta}{2\beta'}.$$

(7) A circular plate is laid upon a smooth table; a snail is placed upon the plate at a given distance from the centre of the plate: supposing the snail to crawl along the plate in a circular path relatively to the plate's centre, to find the motion of the centre of the plate.

If m, m', denote the masses of the plate and snail respectively, and a the radius of the relative circle described by the snail, the centre of the plate will also describe a circle the radius of which is equal to

$$\frac{m'a}{m+m'}.$$

(8) A rigid needle of insensible mass, movable about one end, is held horizontally, a fat fly sitting upon it at a given distance from the fixed end: supposing the needle to be let go, and the fly to run towards the fixed end, so that the needle descends with a uniform angular acceleration, to determine the motion of the fly along the needle.

If r be the distance of the fly from the fixed end of the needle at any time t from the commencement of the motion, a being the initial value of r; then

$$2t\frac{dr}{dt} + r = a \cos\left(\frac{gt^2}{2a}\right).$$

(9) A monkey ascends a ladder with a uniform velocity, the lower end of the ladder being fixed, the higher resting against a vertical wall: to find the pressure of the ladder on the wall.

Let $2a$ denote the length of the ladder, α its inclination to the horizon, x the distance of the monkey, at any time, from the foot of the ladder, and R the corresponding pressure of the ladder on the wall; then, m, m', denoting the respective masses of the ladder and the monkey,

$$R = \frac{g \cot \alpha}{2a}(ma + m'x).$$

(10) A plank, upon which at the upper end a dog is standing, is placed directly along a smooth inclined plane; to determine how long it will take the dog to run down the plank, so that the plank may not stir till he is off it.

If l = the length of the plank, α = the inclination of the plane; then, m, m', being the masses of the plank and dog respectively, the required time is equal to

$$\left\{\frac{2m'l}{g \sin \alpha\,(m + m')}\right\}^{\frac{1}{2}}.$$

(11) A rope of inconsiderable weight is suspended over a smooth pulley: two monkeys commence together clambering up, without jerking, the two portions of the rope, in such a manner that the rope does not slide over the pulley, and that they both reach the pulley at the same moment: to find the time of their ascent to the pulley, their initial positions and their masses being given.

Let m, m', denote the masses of the monkeys, and a, a', their initial distances below the axis of the pulley: then the required time is equal to

$$\left(\frac{2}{g}\right)^{\frac{1}{2}} \cdot \left(\frac{ma - m'a'}{m' - m}\right)^{\frac{1}{2}}.$$

(12) A small beetle, placed at an end of the horizontal diameter of a thin heavy motionless ring, which is movable about its centre in a vertical plane, starts off to crawl round the ring, so as to describe in space equal angles in equal times about its centre: to determine its velocity relatively to the ring in any position, and its pressure on the ring.

Let P (fig. 256) be the position of the beetle at any time t after starting, O the centre of the ring, Ox a horizontal line. Let a = the radius of the ring, m = its mass, m' = the mass of the beetle, $\angle POx = \theta$, ω = the constant value of $\frac{d\theta}{dt}$. Let N, T, denote the normal and tangential pressures of the beetle on the ring.

Then the required relative velocity of the beetle is equal to

$$\frac{m'g}{m\omega} \cdot \sin \omega t + a\frac{m+m'}{m}\omega;$$

also $$N = m'(g \sin \omega t - a\omega^2), \qquad T = m'g \cos \omega t.$$

Mackenzie and Walton; *Solutions of The Cambridge Problems for* 1854.

(13) A circular ring is placed vertically upon a perfectly rough table; and an earwig is put gently upon the ring: to investigate the angular velocity of the earwig in any position about the centre of the ring in order that the ring may not stir.

Let a be the radius of the ring, θ the inclination, to the horizon, of the earwig's distance from the ring's centre at any time, and let α be the initial value of θ. Then, ω being the angular velocity of the earwig in this position,

$$\omega^2 = \frac{g}{a} \cdot \frac{(\sin \alpha - \sin \theta)^2 . (2 + \sin \alpha + \sin \theta)}{(1 + \sin \theta)^2}.$$

(14) A perfectly rough circular hoop rolls with a uniform velocity directly up a perfectly rough inclined plane by the action of a mouse running along its circumference: to determine the angular velocity of the mouse, in any position, about the centre of the hoop, the angular velocity of the mouse in a given position being known.

Let C (fig. 257) be the centre of the hoop at any time, H the point of contact of the hoop and the inclined plane AB, P the position of the mouse. Let $\angle PCH = \theta$, $\omega =$ the angular velocity of the mouse about C, $m =$ the mass of the hoop, $m' =$ the mass of the mouse, $\alpha =$ the inclination of AB to the horizon, $a =$ the radius of the hoop: then

$$m'a(1-\cos\theta)^2\omega^2 = C + 2g(m+m')\sin\alpha\,(\theta - \sin\theta)$$
$$+ 2m'g\{\cos(\theta+\alpha) - \cos(2\theta+\alpha) + 2\theta\sin\alpha\},$$

C being a constant, which may be expressed in terms of given simultaneous values of ω and θ.

(15) A rod AB (fig. 258) attached to a horizontal rod Ox and a vertical rod Oy by rings at its ends, is kept at rest by a man's hand, while a monkey is sitting upon a small platform C vertically above O: the monkey then springs horizontally from C and alights on the middle point G of the rod, to which it clings tightly: to determine the motion of the rod the instant after the monkey's arrival at G, supposing the man to have loosed his hold the instant before.

Let $AG = a = BG$, $\angle ABO = \theta$, $m =$ the mass of the movable rod, $m' =$ the mass of the monkey, $u =$ the velocity with which the monkey springs from C, $h =$ the vertical altitude of C above G: then the angular velocity of AB, the instant after the monkey's arrival at G, is equal to

$$\frac{3m'}{a} \cdot \frac{(2gh)^{\frac{1}{2}}\cos\theta + u\sin\theta}{4m+3m'}.$$

CHAPTER XIV.

MISCELLANEOUS PROBLEMS.

(1) A PERFECTLY elastic ball is thrown into a smooth cylindrical well from a point in the circumference of the circular mouth: shew that, if the ball be reflected any number of times from the surface of the cylinder, the intervals between the reflections will be equal: shew also that, if the ball be projected horizontally in a direction making an angle $\frac{\pi}{n}$ with the tangent at the point of projection, it will reach the surface of the water at the instant of the n^{th} reflection, if the space due to the velocity of projection be equal to

$$\frac{(\text{radius})^2}{\text{depth}} \times \left(n \sin \frac{\pi}{n}\right)^2.$$

(2) A perfectly elastic ball impinges with a velocity v and at an angle α to the horizon, on an inclined plane: the direction of impact is in a vertical plane parallel to the plane's intersection with the horizon: after rebounding it falls on this line of intersection: shew that

$$2v \sin \alpha \sin \lambda = (gh)^{\frac{1}{2}},$$

λ being the plane's inclination to the horizon, and h the distance of the first point of impact from the horizontal plane.

(3) A ball, thrown from any point in one of the walls of a rectangular room, returns, after striking the three others, to the point of projection, before it falls to the ground: shew that the space due to the velocity of projection is greater than the diagonal of the floor.

(4) A body, acted on by gravity, is projected from a given point; and, when it has reached its greatest height, another body

is projected from the same point in such a manner that it shall strike the first body: shew that

$$\frac{u'}{u} - 2\frac{v'}{v} = 1.$$

(5) If two heavy particles be projected, with equal velocities from the extremity of the vertical diameter of a vertical circle, the one down the diameter, the other along any chord terminating in that extremity, prove that the chord will be described in a less time than the diameter.

(6) If a particle be projected from one extremity of the axis major of an ellipse, the minor axis of which is vertical, so as to exactly shoot down a thin straight tube extending from the upper end of the minor axis to the other end of the axis major; shew that, α being the angle of projection, and $\cos\beta$ the eccentricity of the ellipse,

$$\tan\alpha = 3\sin\beta.$$

(7) *ABCDE* being a regular pentagon, placed in a vertical plane, with the lower side *CD* horizontal, the times in which a particle would descend down *BC*, *AC*, *EC*, respectively, are in geometrical progression.

(8) A body, hanging vertically, drew another body of half the weight, by means of a fine string, up an inclined plane: when the bodies had described a space c, the string broke, and it was found that the smaller body continued its motion in the same direction through an additional space c before it began to descend: to find the inclination of the plane.

$$\text{The required inclination} = \frac{\pi}{6}.$$

(9) A body, descending vertically, draws an equal body 25 feet in $2\frac{1}{2}$ seconds up a plane inclined at 30° to the horizon, by means of a fine string passing over a pulley at the top of the plane: to determine the force of gravity.

Gravity = 32 feet.

(10) A heavy body P, descending vertically, draws another body Q through a space of $16\frac{1}{10}$ feet in 4 seconds up a plane inclined to the horizon at an angle of $30°$, by means of a fine string passing over a pulley at the top of the plane: shew that

$$P : Q :: 3 : 5.$$

(11) If the product of the velocities at the two points P, Q, of the parabolic path of a body, acted on by gravity, be constant, shew that the locus of the pole of PQ is a circle, having the focus of the parabola for its centre.

(12) If a circle be described on the transverse axis (supposed to be vertical) of an equilateral hyperbola, the sum of the squares of the times in which a molecule would fall down two lines drawn from a point in the circle to the extremities of a given double ordinate of the hyperbola, will be the same for every point in the circle.

(13) A particle is placed on the surface of an ellipsoid, in the centre of which resides an attractive force: to determine the direction in which the particle will begin to move.

If α, β, γ, be the place of the particle on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the particle will begin to move in the line of which the equations are

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

and
$$\frac{x}{\alpha}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + \frac{y}{\beta}\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + \frac{z}{\gamma}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0.$$

(14) Two ships are sailing uniformly with velocities u, v, along lines inclined at an angle θ: shew that, if a, b, be their distances at one time from the point of intersection of the courses, the least distance of the ships is equal to

$$\frac{(av - bu)\sin\theta}{(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}}.$$

(15) In an ellipse of small eccentricity, the centre of force being in the focus, the equation of the centre varies nearly as the velocity parallel to the axis major.

(16) A particle, acted on by a central force P, is moving in a medium the resistance of which varies as the velocity: prove that

$$\frac{d^2r}{dt^2} + P - \frac{h^2}{r^3} \cdot e^{-2ct} + c\frac{dr}{dt} = 0,$$

where c and h are constants.

(17) A horizontal tube revolves uniformly about a vertical axis and a heavy particle is placed in it always at a given distance from this axis: the particle is shot out when the tube comes into a given position and falls on a horizontal plane: shew that, if the length of the tube be supposed to vary, the angular velocity remaining the same, the assemblage of points of impact on the plane will form a conic section, but that, when the angular velocity varies, the length of the tube being constant, the locus is a straight line.

(18) A bead is enclosed in a smooth circular tube the centre of which moves uniformly in a straight line in the plane of the tube: shew that the velocity of the bead relatively to the tube is uniform.

(19) If P be the perimeter of a closed curve described about a centre of force, τ the time of revolution, h twice the area described in a unit of time, and ρ the radius of curvature at the time t, prove that

$$P = h\int_0^\tau \frac{dt}{\rho}.$$

(20) A smooth rod is made to revolve uniformly with an angular velocity ω in a vertical plane about a pivot through one extremity: a heavy particle is placed on the rod when it is in a horizontal position, at a distance a from the point: shew that, if ω be very small, the particle will reach the pivot in a time equal to $\left(\frac{6a}{g\omega}\right)^{\frac{1}{3}}$.

(21) A bead is placed on a smooth circular wire, which receives a given angular velocity in its own plane about a point of its circumference: prove that the initial velocity of the bead is half that of a point of the wire the distance of which from the bead is equal to the bead's from the fixed point.

(22) If the circle which generates a cycloid roll along the directrix, supposed horizontal, with a certain uniform velocity, shew that the velocity of the describing point is in each position the same as if it had slid, acted on by gravity and without friction, along the cycloid from rest at a cusp. (The vertex of the cycloid lies below the directrix.)

(23) A circular hoop of radius a revolves with a uniform angular velocity ω round a vertical diameter, and a smooth heavy ring slides upon it: shew that, when $\omega < \left(\frac{g}{a}\right)^{\frac{1}{2}}$, the ring may perform small oscillations of the period

$$\frac{\pi}{\left(\frac{g}{a} - \omega^2\right)^{\frac{1}{2}}}$$

about the lowest point.

(24) A particle is constrained to move in a circle and is acted on by a force tending to a fixed point and varying inversely as the distance: prove that the sum of the squares of the velocities of the particle at the extremities of any chord drawn through the centre of force is constant.

(25) A material particle, placed at a centre of attraction varying as the distance, is urged from rest by a constant force, which acts for one sixth of the time of a complete oscillation about the centre, ceases for the same period, and then acts as before: shew that the particle will then be retained at rest, and that the spaces moved through in the two periods are equal.

(26) The inclination of a smooth inclined plane is such that the pressure on it of a body, supported by a string parallel to the plane, is increased by one half when the string is cut and the

body supported by moving the plane horizontally: shew that, if the plane be stopped suddenly at any time after the commencement of the motion, the body will strike it again after an equal interval.

(27) A particle, acted on by no forces, moves in a rough groove in one plane: shew that the difference between the logarithms of the velocities at any two points varies as the angle between the tangents at those points.

(28) If the arc of a circle containing an angle $\cot^{-1}(-\mu)$ be placed so that its bounding chord is vertical: then the times of descent down all chords considered rough, which are drawn from the highest point, will be equal, μ being the coefficient of friction.

(29) A particle, attracted towards a fixed centre, the force varying as the distance, is projected with a given velocity from a given point: supposing the locus of the direction of projection to be a plane, the locus of the orbit is an ellipsoid, of which the particle's original position is an umbilical point.

(30) A tube of small bore, in the form of a logarithmic spiral, revolves with a uniform angular velocity about an axis passing through its pole and perpendicular to its plane, which is horizontal, and contains a particle which moves freely in it; supposing the initial velocity of the particle relatively to the tube to be equal to the velocity of the point of the tube in contact with the particle, shew that the path of the particle is another logarithmic spiral.

(31) Shew that a boat must be rowed with a velocity, through the water, one half greater than that of the stream, so that it may be taken a given distance up a river, with the least possible expenditure of work. (The resistance to the boat is supposed to be proportional to the square of its velocity through the water.)

(32) A particle slides down a smooth inclined plane which moves with uniformly accelerated velocity directly forwards on

a horizontal plane: the particle and plane begin to move at the same instant: shew that, if v be the velocity of the particle relatively to the plane, $v^2 = 2g's$, where s is the space described along the plane, and

$$g' = g \sin \alpha - u \cos \alpha,$$

α being the inclination of the plane and u its horizontal acceleration.

(33) Two heavy particles are connected by a fine inelastic string, which is just stretched, and one of them is struck in a direction perpendicular to the string: shew that they will never approach each other.

(34) The two parts m, m', of a compound vibrating body have their centres of gravity H, H', centres of oscillation O, O', and point of connection P in a straight line passing through S the axis of suspension: $SH = h$, $SH' = h'$, $SO = l$, $SO' = l'$, $SP = e$, and r, r', are the respective expansions of linear units of the masses for a given rise of temperature. Shew, first, that the change of l' for the given change of temperature is

$$r'l' + e(r - r')\left(2 - \frac{l'}{h'}\right);$$

and then that the length L of the compound pendulum is unaltered by temperature if

$$\tfrac{1}{2}L = \frac{mrhl + m'r'h'l' + m'h'e(r - r')}{mrh + m'r'h' + m'e(r - r')}.$$

(35) When a circular lamina, the plane of which is vertical, rolls along a horizontal plane, shew, by calculating the value of each, that the diminution of pressure arising from the centrifugal force is compensated by the increase of pressure arising from the effective force on each particle in the direction of its motion.

(36) Shew that if one of a free system of particles, which attract one another with forces varying as the mass and distance, have at any instant the same position and the same velocity, in magnitude and direction, as the centre of gravity of the system, it will coincide with it throughout the motion.

(37) Two solid globes, each of radius a and density Δ, attracting by the force of gravity two minute spheres at the extremities of the arms of a torsion balance in directions perpendicular to the arms and at a distance b from their centres, were observed to retain the spheres at a distance s from their position of rest; and the time of a free oscillation of the balance was found to be t seconds: shew that, if L be the length of a seconds pendulum, and R the Earth's radius, the density of the Earth is equal to

$$\frac{La^3t^2\Delta}{Rb^2s}, \text{ nearly.}$$

(38) A slender rod, suspended horizontally by two equal strings attached to two points equidistant from its ends, performs small oscillations round a vertical line through its middle point: prove that, if in the position of equilibrium the strings are inclined at equal angles to the vertical, the time of oscillation is the same as it would be if the strings were parallel, of a length equal to the projection of either of them on a vertical line, and at a distance equal to a mean proportional between their distances at the points of suspension and attachment respectively.

(39) The form of a homogeneous solid of revolution, of given superficial area, described upon an axis of given length, is such that its moment of inertia about the axis is a maximum: prove that the normal at any point of the generating curve is three times as long as the radius of curvature.

Mackenzie and Walton; *Solutions of the Cambridge Problems for* 1854.

(40) When a flat body, resting upon a horizontal plane, receives a blow, shew that, in the time which the system takes to make a complete revolution, the centre of gravity will advance over a space equal to the circumference of the circle described about the spontaneous centre of rotation as centre and passing through the centre of gravity.

(41) Two rigid bodies move, acted on by no force, about fixed points: the principal moments of inertia of the one about its fixed point are in the duplicate ratio of the corresponding

moments of the other: prove that the initial circumstances may be so adjusted that, if the angles which the instantaneous axis makes with the corresponding principal axes are at any instants equal each to each in the two bodies, their angular velocities are also equal.

(42) A circular plate revolves about its centre of gravity, which is fixed: if ω be the angular velocity about the instantaneous axis at the commencement of the motion and this axis then make an angle α with the plane, the instantaneous axis will make a complete revolution in the time

$$\frac{2\pi}{\omega\,(1+3\sin^2\alpha)^{\frac{1}{2}}}.$$

Griffin; *Solutions of the Examples appended to a Treatise on the Motion of a Rigid Body.*

(43) Dato Pendulo turbinante, composito ex ponderibus non in communi turbinationis plano, sed vel in alio, vel in aliis diversis planis inhærentibus; demissisque rectis perpendicularibus ad commune planum turbinationis ex ponderibus; si pondera singula ducantur in distantias suarum perpendicularium ab axe turbinationis et porro in altitudines superficierum conicarum, quas rectæ a ponderibus ad verticem turbinationis eductæ describunt; deinde summa productorum dividatur per id, quod fit ducendo ponderum summam in distantiam centri gravitatis communis omnium ab axe turbinationis: habebitur distantia Centri turbinationis, seu longitudo Penduli simplicis circuitus minimos iisdem cum composito temporibus facientis, sive altitudo superficiei conicæ, quam Pendulum quodlibet simplex describens Pendulo dato composito erit isochronum[1].

John Bernoulli; *Acta Erud. Lips.* 1715. Jun. pag. 242. *Opera, Tom.* II. p. 187.

[1] Let G (fig. 259) be the centre of gravity of any rigid body, acted on by gravity, and revolving conically about a fixed point C to which it is fixed, the path of each of its particles being a horizontal circle. Let CO be a vertical line. The body is called a *turbinating pendulum*, C the *vertex of turbination*, CO the *axis of turbination*, the plane OCG the *plane of turbination*; the *centre of turbination* of the compound pendulum is a point in the axis of turbination at a distance from the vertex of turbination equal to the altitude of the conical superficies described by a simple pendulum turbinating isochronously with the compound pendulum. A simple pendulum is said to make *circuitus minimos* when the conical angle is indefinitely small.

APPENDIX.

THE COPY of Bernoulli's programme[1] which had been received by the celebrated David Gregory, was seen some years ago by the author of this work, in the possession of the lamented Mr. Gregory, late Fellow of Trinity College. The following reprint of the challenge will probably be acceptable to those who take an interest in the antiquities of science.

Acutissimis qui toto orbe florent Mathematicis.

S. P. D.

JOHANNES BERNOULLI, MATH. P.P.

"Cum compertum habeamus, vix quicquam esse quod magis excitet generosa ingenia ad moliendum quod conducit augendis scientiis, quàm difficilium pariter et utilium quæstionum propositionem, quarum enodatione tanquam singulari si qua aliâ via ad nominis claritatem perveniant sibique apud posteritatem æterna extruant monumenta: Sic me nihil gratius Orbi Mathematico facturum speravi quam si imitando exemplum tantorum Virorum Mersenni, Pascalii, Fermatii, præsertim recentis illius Anonymi Ænigmatistæ Florentini, aliorumque qui idem ante me fecerunt, præstantissimis hujus ævi Analystis proponerem aliquod problema, quo quasi Lapide Lydio suas methodos examinare, vires intendere et si quid invenirent nobiscum communicare possent, ut quisque suas exinde promeritas laudes à nobis publicè id profitentibus consequeretur.

"Factum autem illud est ante semestre in Actis Lips. m. Jun. pag. 269. Ubi tale problema proposui cujus utilitatem cum jucunditate conjunctam videbunt omnes qui cum successu ei se applicabunt. Sex mensium spatium à prima publicationis die Geometris concessum est, intra quod si nulla solutio prodiret in lucem, me meam exhibiturum promisi: Sed ecce elapsus est

[1] See page 312.

terminus et nihil solutionis comparuit; nisi quod Celeb. Leibnitius de profundiore Geometriâ præclarè meritus me per literas certiorem fecerit, se jam feliciter dissolvisse nodum pulcherrimi hujus uti vocabat et inauditi antea problematis, insimulque humaniter rogavit, ut præstitutum limitem ad proximum pascha extendi paterer, quo interea apud Gallos Italosque idem illud publicari posset nullusque adeo superesset locus ulli de angustiâ termini querelæ; Quam honestam petitionem non solum indulsi, sed ipse hanc prorogationem promulgare decrevi, visurus num qui sint qui nobilem hanc et arduam quæstionem aggressuri, post longum temporis intervallum tandem Enodationis compotes fierent. Illorum interim in gratiam ad quorum manus Acta Lipsiensia non perveniunt, propositionem hic repeto.

PROBLEMA MECHANICO-GEOMETRICUM DE LINEA CELERRIMI DESCENSUS.

"*Determinare lineam curvam data duo puncta in diversis ab horizonte distantiis et non in eadem rectâ verticali posita connectentem, super qua mobile propriâ gravitate decurrens et â superiori puncto moveri incipiens citissime descendat ad punctum inferius.*

"Sensus problematis hic est, ex infinitis lineis quæ duo illa data puncta conjungunt, vel ab uno ad alterum duci possunt eligatur illa, juxta quam si incurvetur lamina tubi canalisve formam habens, ut ipsi impositus globulus et liberè dimissus iter suum ab uno puncto ad alterum emetiatur tempore brevissimo.

"Ut vero omnem ambiguitatis ansam præcaveamus, scire B.L. volumus, nos hîc admittere Galilæi hypothesin de cujus veritate sepositâ resistentiâ jam nemo est saniorum Geometrarum qui ambigat, *Velocitates scilicet acquisitas gravium cadentium esse in subduplicata ratione altitudinum emensarum,* quanquam aliàs nostra solvendi methodus universaliter ad quamvis aliam hypothesin sese extendat.

"Cum itaque nihil obscuritatis supersit, obnixè rogamus omnes et singulos hujus ævi Geometras, accingant se promtè, tentent, discutiant quicquid in extremo suarum methodorum

recessu absconditum tenent; Rapiat qui potest præmium quod Solutori paravimus, non quidem auri non argenti summam quo abjecta tantum et mercenaria conducuntur ingenia, à quibus ùt nihil laudabile sic nihil quod scientiis fructuosum expectamus, sed cùm virtus sibi ipsi sit merces pulcherrima, atque gloria immensum habeat calcar, offerimus præmium quale convenit ingenui sanguinis Viro, consertum ex honore, laude et plausu, quibus magni nostri Apollinis perspicacitatem publicè et privatim, scriptis et dictis coronabimus, condecorabimus et celebrabimus.

"Quod si verò festum paschatis præterierit nemine deprehenso qui quæsitum nostrum solverit, nos quæ ipsi invenimus publico non invidebimus; Incomparabilis enim Leibnitius solutiones tum suam tum nostram ipsi jam pridem commissam protinus ut spero in lucem emittet, quas si Geometræ ex penitiori quodam fonte petitas perspexerint, nulli dubitamus quin angustos vulgaris Geometriæ limites agnoscant, nostraque proin inventa tanto pluris faciant, quanto pauciores eximiam nostram quæstionem soluturi extiterint etiam inter illos ipsos qui per singulares quas tantopere commendant methodos, interioris Geometriæ latibula non solum intimè penetrâsse, sed etiam ejus pomœria Theorematis suis aureis, nemini ut putabant cognitis, ab aliis tamen jam longè priùs editis mirum in modum extendisse gloriantur.

PROBLEMA ALTERUM PURE GEOMETRICUM, QUOD PRIORI SUBNECTIMUS ET STRENÆ LOCO ERUDITIS PROPONIMUS.

"Ab Euclidis tempore vel Tyronibus notum est; Ductam utcunque à puncto dato rectam lineam, à circuli peripheriâ ita secari ut rectangulum duorum segmentorum inter punctum datum et utramque peripheriæ partem interceptorum sit eidem constanti perpetuo æquale. Primus ego ostendi in eod. Actor. Jun. pag. 265. hanc proprietatem infinitis aliis curvis convenire, illamque adeo circulo non esse essentialem. Arrepta hinc occasione, proposui Geometris determinandam curvam vel curvas, in quibus non rectangulum sed solidum sub uno et quadrato alterius segmentorum æquetur semper eidem; sed à nemine hactenus solvendi modus prodiit; exhibebimus eum quandocunque desi-

derabitur: Quoniam autem non nisi per curvas transcendentes quæsito satisfacimus, en aliud cujus solutio per merè algebricas in nostra est potestate. *Quæritur Curva, ejus proprietatis, ut duo illa segmenta ad quamcunque potentiam datam elevata et simul sumta faciant ubique unam eandemque summam.*

"Casum simplicissimum existente sc. numero potentiæ 1. ibidem in actis pag. 266. jam solutum dedimus, generalem verò solutionem quam etiamnum premimus, Analystis eruendam relinquimus."

Dabam Groningæ ipsis Cal. Jan. 1697.

Groningæ, Typis Catharinæ Zandt, Provincialis Academiæ Typographæ, 1697.

THE END.

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.

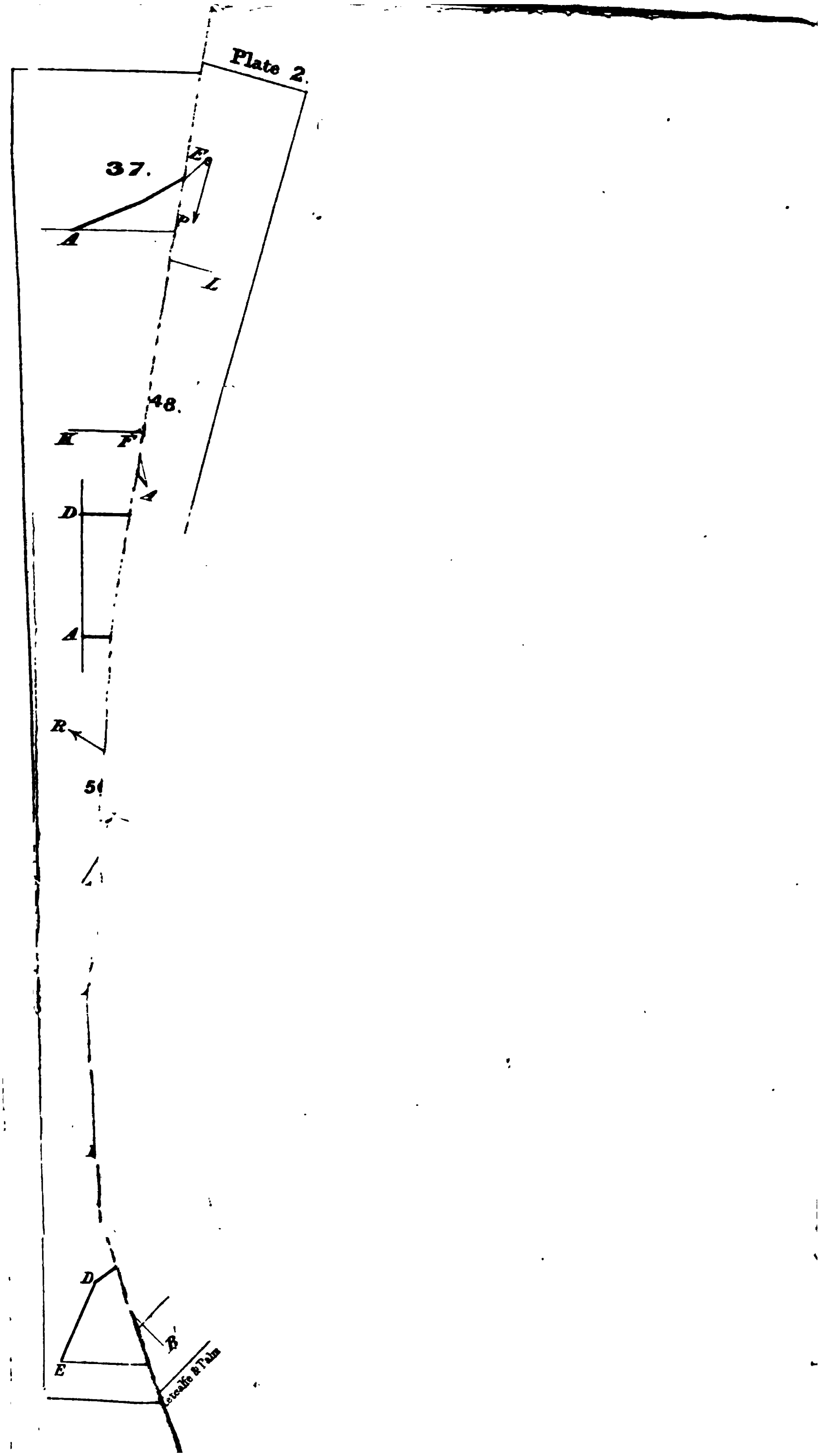
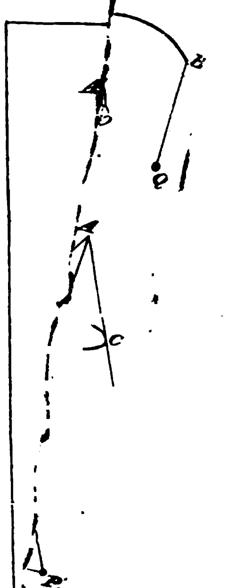
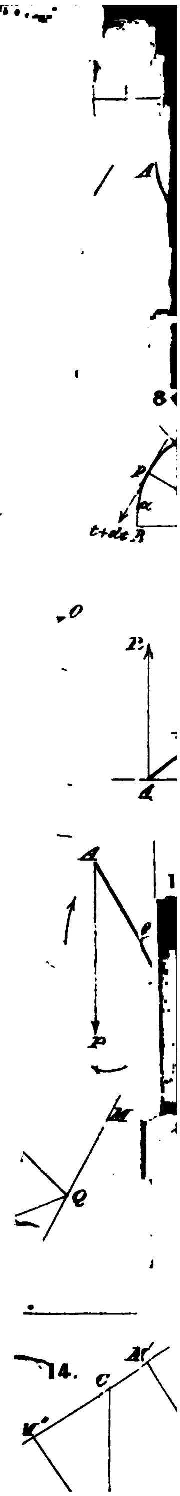
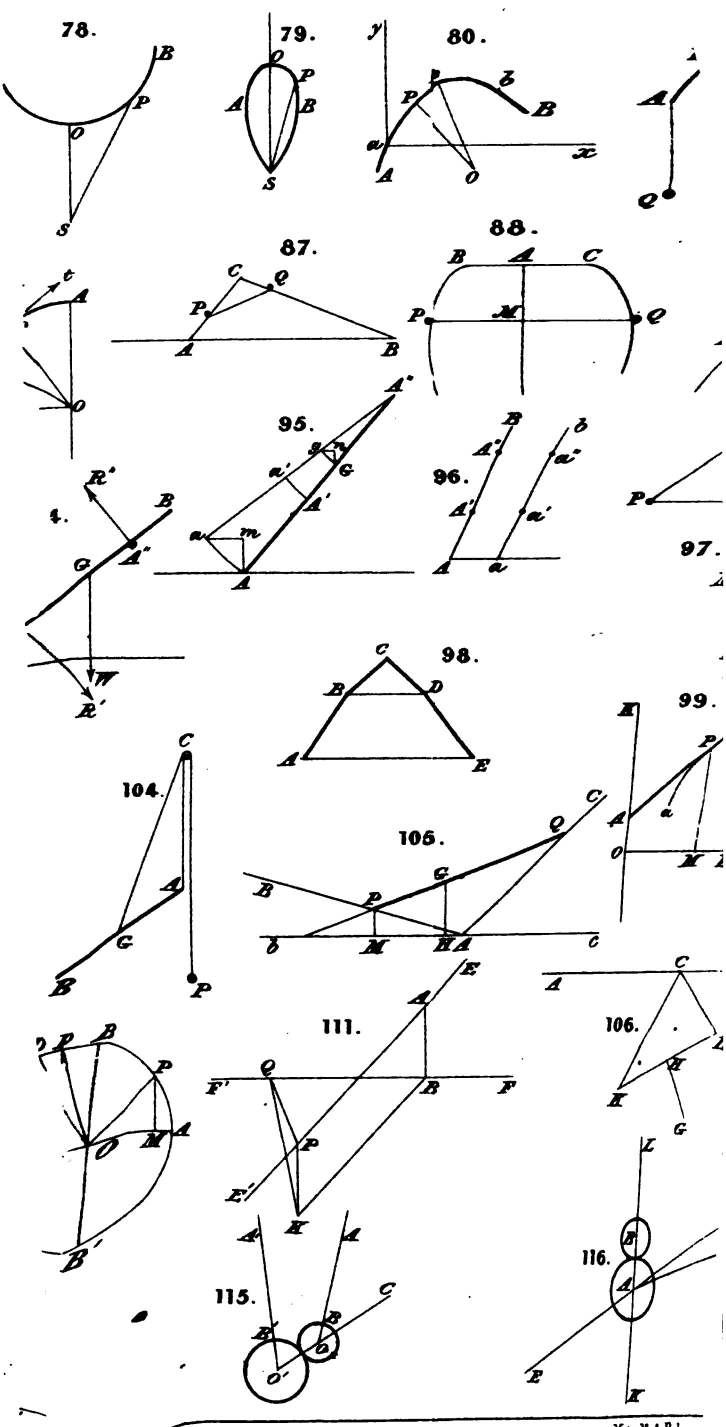


Plate 2.
37.
48.
Metcalfe & Palm

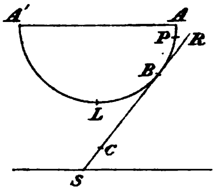
Plate

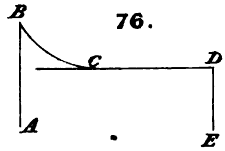


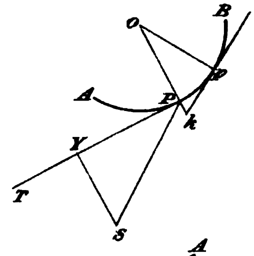


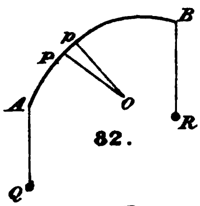


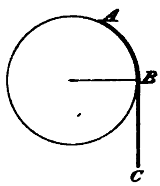
Metcalfe & Palmer,


75.

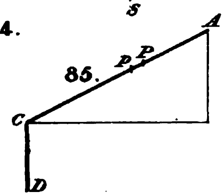

76.

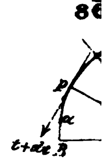

77.

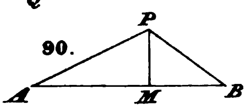

82.

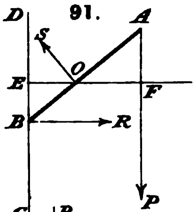

83.

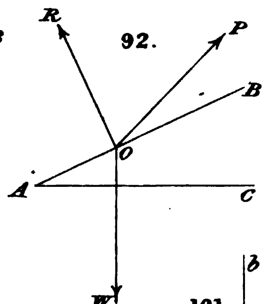

84.

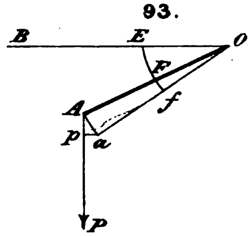

85.

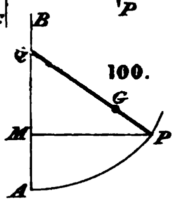

86

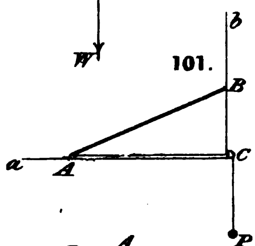

90.

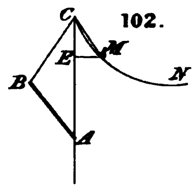

91.

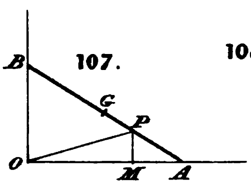

92.

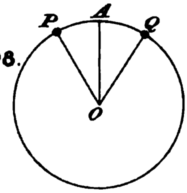

93.

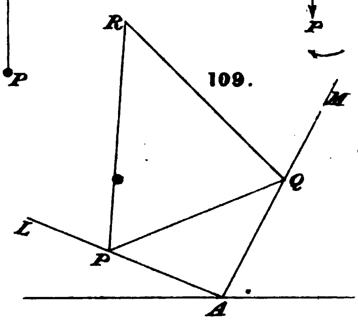

100.

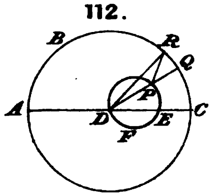

101.

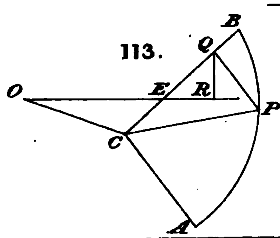

102.

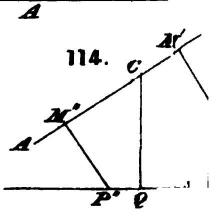

107.

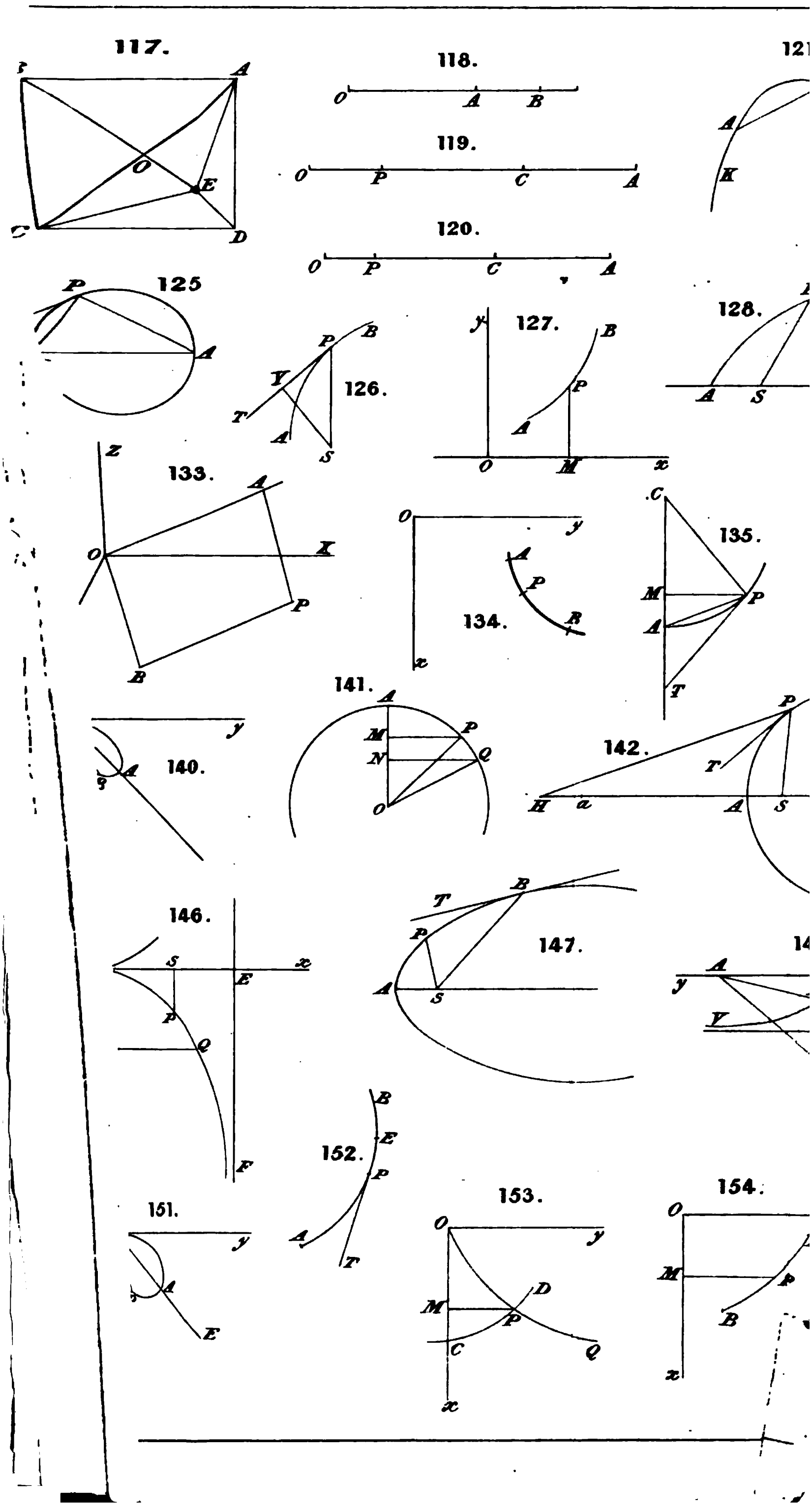

108.

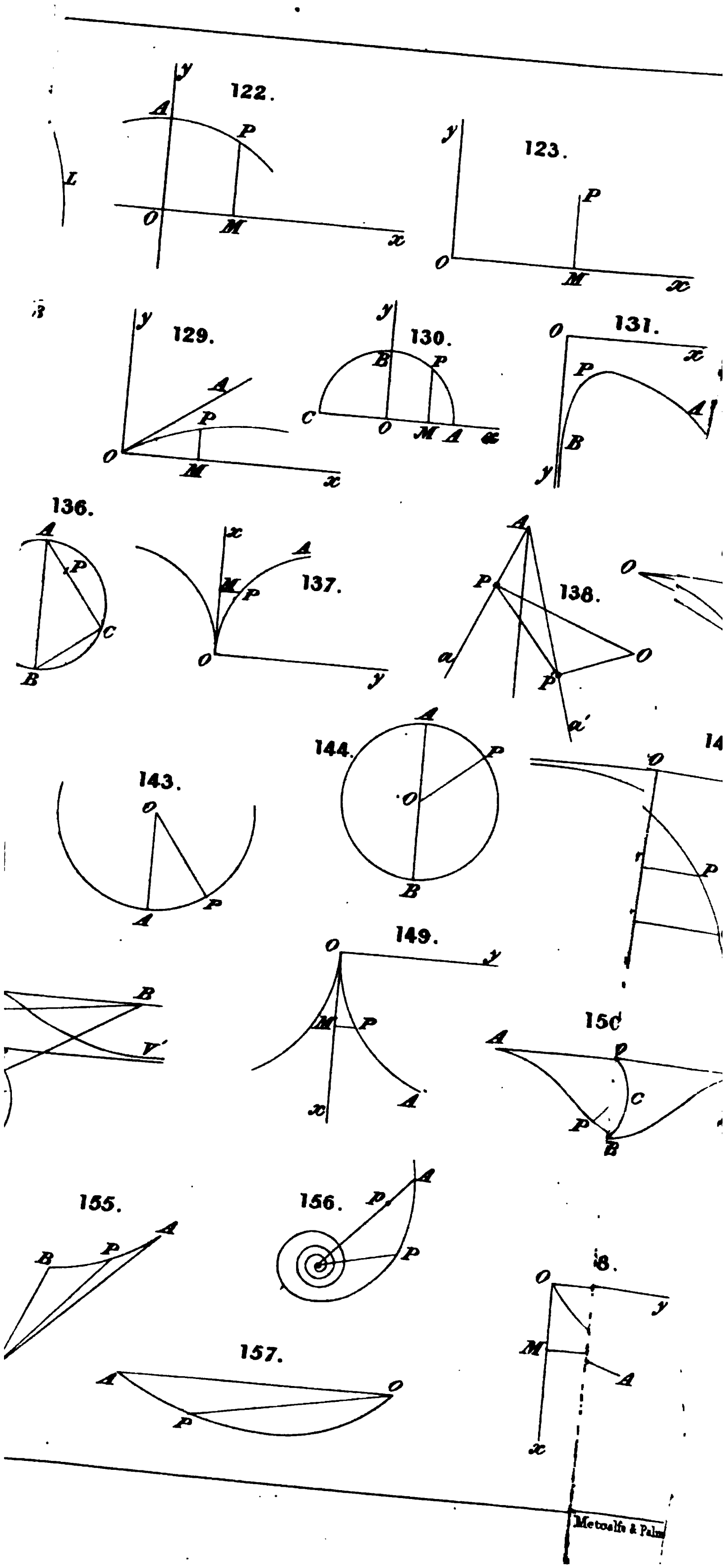

109.


112.

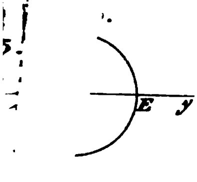

113.


114.

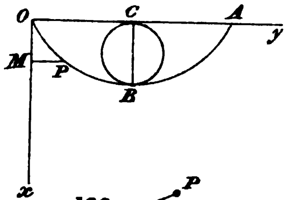

117.
118.
119.
120.
121
125
126.
127.
128.
133.
134.
135.
140.
141.
142.
146.
147.
14
151.
152.
153.
154.



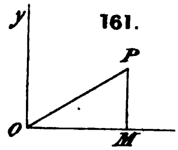
122.
123.
129.
130.
131.
136.
137.
138.
143.
144.
149.
155.
156.
157.
Metcalfe & Palm



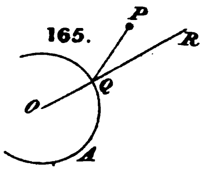
160.



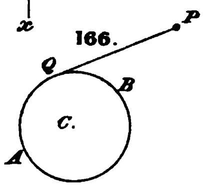
161.



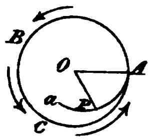
165.



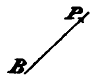
166.



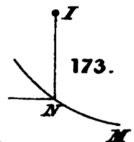
167.



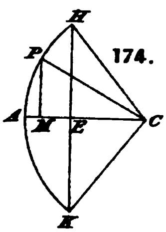
168.



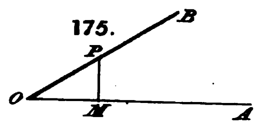
173.



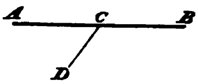
174.



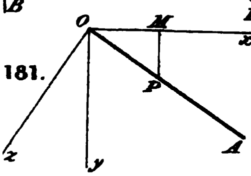
175.



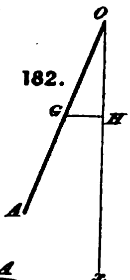
176.



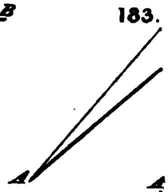
181.



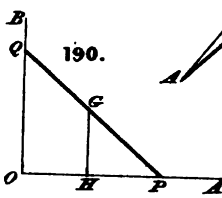
182.



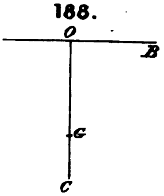
183.



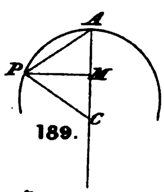
190.



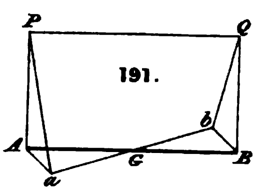
188.



189.

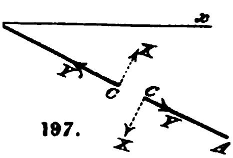


191.

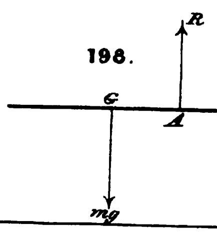


19:

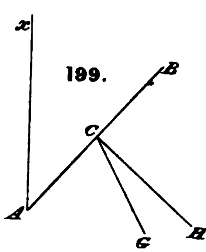
197.



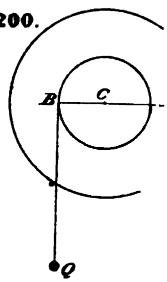
198.

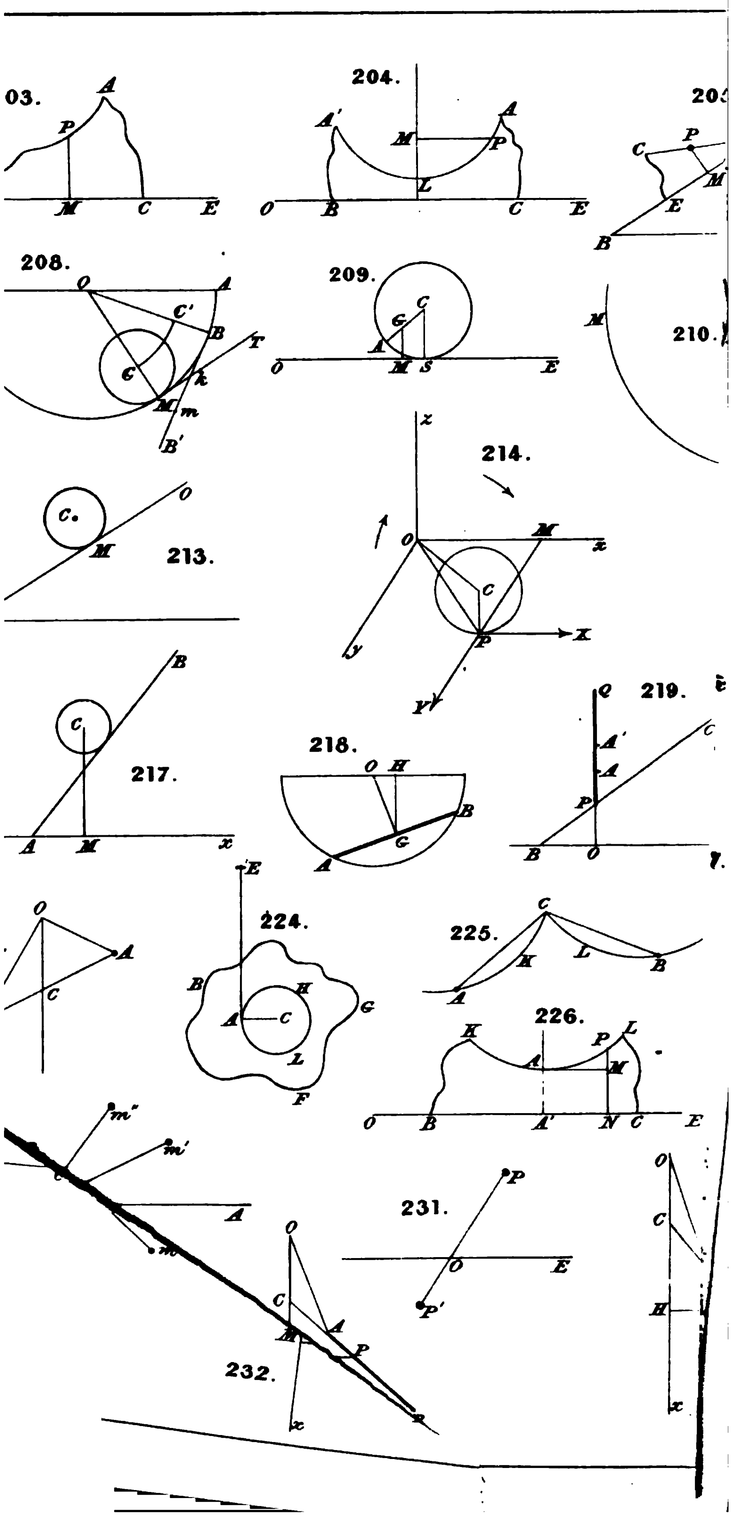


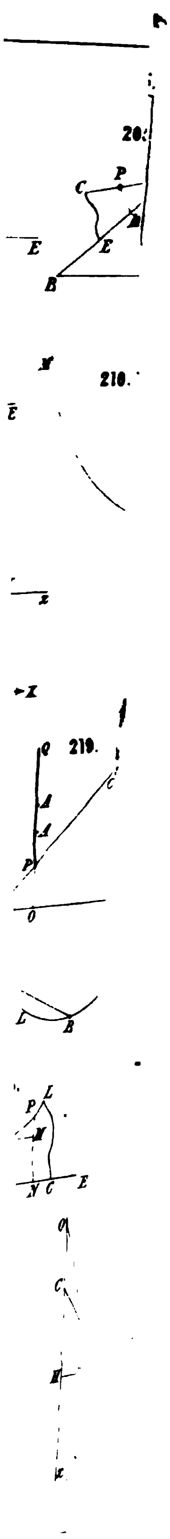
199.



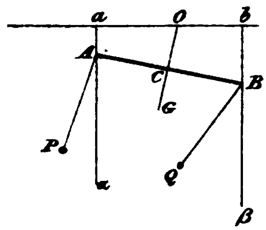
200.



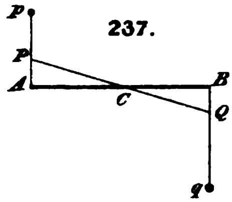

03.
204.
20
208.
209.
210.
213.
214.
217.
218.
219.
224.
225.
226.
231.
232.


C
P
E
B
210.
219.
Q
P
O
L
B
P
L
N
C
E

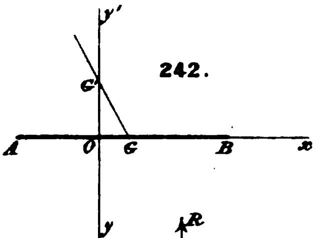
236.



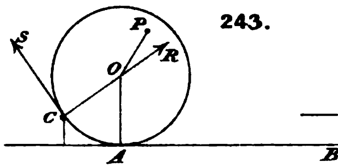
237.



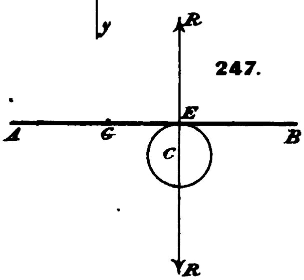
242.



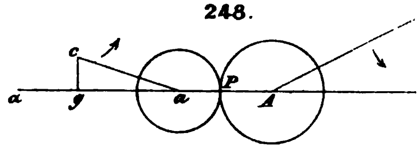
243.



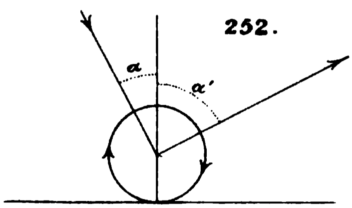
247.



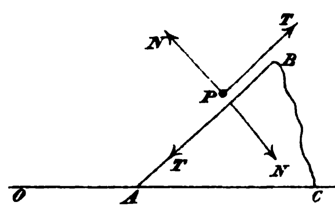
248.



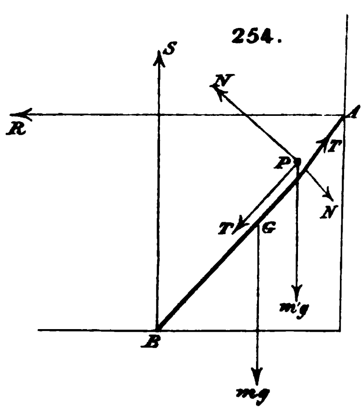
252.



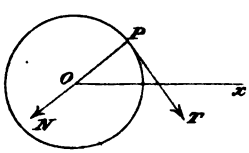
25.

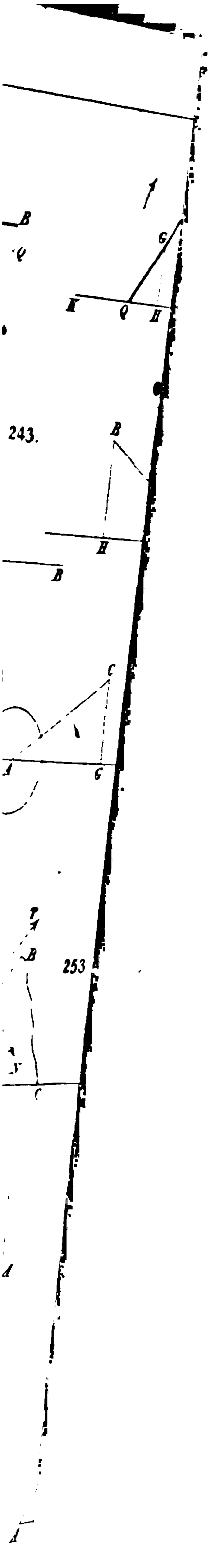


254.



256.





243.

253